We obtain this estimate by interpolation. Obviously (7.1.28) is valid when $q=\infty$ with $C(\infty,n)=1$. If we prove that

$$\big\|\mathcal{M}_c^w\big\|_{L^1(w)\to L^{1,\infty}(w)} \le C(1,n)<\infty\,, \tag{7.1.29}$$

then (7.1.28) will follow from Theorem 1.3.2.

To prove (7.1.29) we fix $f\in L^1(\mathbf{R}^n,w\,dx)$. We first show that the set

$$E_\lambda=\{\mathcal{M}_c^w(f)>\lambda\}$$

is open. For any $r>0$, let $Q(x,r)$ denote an open cube of side length $2r$ with center $x\in\mathbf{R}^n$. If we show that for any $r>0$ and $x\in\mathbf{R}^n$ the function

$$x\mapsto \frac{1}{w(Q(x,r))}\int_{Q(x,r)}|f|\,w\,dy \tag{7.1.30}$$

is continuous, then $\mathcal{M}_c^w(f)$ is the supremum of continuous functions; hence it is lower semicontinuous and thus the set E_λ is open. But this is straightforward. If $x_n\to x_0$, then $w(Q(x_n,r))\to w(Q(x_0,r))$ and also $\int_{Q(x_n,r)}|f|\,w\,dy\to\int_{Q(x_0,r)}|f|\,w\,dy$ by the Lebesgue dominated convergence theorem. Since $w(Q(x_0,r))\neq 0$, it follows that the function in (7.1.30) is continuous.

Given K a compact subset of E_λ, for any $x\in K$ select an open cube Q_x centered at x such that

$$\frac{1}{w(Q_x)}\int_{Q_x}|f|\,w\,dy>\lambda\,.$$

Applying Lemma 7.1.10 (proved immediately afterward) we find a subfamily $\{Q_{x_j}\}_{j=1}^m$ of the family of the cubes $\{Q_x:\ x\in K\}$ such that (7.1.31) and (7.1.32) hold. Then

$$w(K)\le\sum_{j=1}^m w(Q_{x_j})\le\sum_{j=1}^m\frac{1}{\lambda}\int_{Q_{x_j}}|f|\,w\,dy\le\frac{72^n}{\lambda}\int_{\mathbf{R}^n}|f|\,w\,dy\,,$$

where the last inequality follows by multiplying (7.1.32) by $|f|w$ and integrating over $\mathbf{R}^n$. Taking the supremum over all compact subsets K of E_λ and using the inner regularity of $w\,dx$, which is a consequence of the Lebesgue monotone convergence theorem, we deduce that $\mathcal{M}_c^w$ maps $L^1(w)$ to $L^{1,\infty}(w)$ with constant at most 72^n. Thus (7.1.29) holds with $C(1,n)=72^n$. □

Lemma 7.1.10. *Let K be a bounded set in $\mathbf{R}^n$ and for every $x\in K$, let Q_x be an open cube with center x and sides parallel to the axes. Then there exist $m\in\mathbf{Z}^+\cup\{\infty\}$ and a sequence of points $\{x_j\}_{j=1}^m$ in K such that*

$$K\subseteq\bigcup_{j=1}^m Q_{x_j} \tag{7.1.31}$$

and for almost all $y \in \mathbf{R}^n$ one has

$$\sum_{j=1}^{m} \chi_{Q_{x_j}}(y) \le 72^n . \tag{7.1.32}$$

Proof. Let $s_0 = \sup\{\ell(Q_x) : x \in K\}$. If $s_0 = \infty$, then there exists $x_1 \in K$ such that $\ell(Q_{x_1}) > 4L$, where $[-L,L]^n$ contains K. Then K is contained in Q_{x_1} and the statement of the lemma is valid with $m = 1$.

Suppose now that $s_0 < \infty$. Select $x_1 \in K$ such that $\ell(Q_{x_1}) > s_0/2$. Then define

$$K_1 = K \setminus Q_{x_1}, \qquad s_1 = \sup\{\ell(Q_x) : x \in K_1\},$$

and select $x_2 \in K_1$ such that $\ell(Q_{x_2}) > s_1/2$. Next define

$$K_2 = K \setminus (Q_{x_1} \cup Q_{x_2}), \qquad s_2 = \sup\{\ell(Q_x) : x \in K_2\},$$

and select $x_3 \in K_2$ such that $\ell(Q_{x_3}) > s_2/2$. Continue until the first integer m is found such that K_m is an empty set. If no such integer exists, continue this process indefinitely and set $m = \infty$.

We claim that for all $i \neq j$ we have $\frac{1}{3}Q_{x_i} \cap \frac{1}{3}Q_{x_j} = \emptyset$. Indeed, suppose that $i > j$. Then $x_i \in K_{i-1} = K \setminus (Q_{x_1} \cup \cdots \cup Q_{x_{i-1}})$; thus $x_i \notin Q_{x_j}$. Also $x_i \in K_{i-1} \subsetneqq K_{j-1}$, which implies that $\ell(Q_{x_i}) \le s_{j-1} < 2\ell(Q_{x_j})$. Since $x_i \notin Q_{x_j}$ and $\ell(Q_{x_j}) > \frac{1}{2}\ell(Q_{x_i})$, it easily follows that $\frac{1}{3}Q_{x_i} \cap \frac{1}{3}Q_{x_j} = \emptyset$.

We now prove (7.1.31). If $m < \infty$, then $K_m = \emptyset$ and therefore $K \subsetneqq \bigcup_{j=1}^{m} Q_{x_j}$. If $m = \infty$, then there is an infinite number of selected cubes Q_{x_j}. Since the cubes $\frac{1}{3}Q_{x_j}$ are pairwise disjoint and have centers in a bounded set, it must be the case that some subsequence of the sequence of their lengths converges to zero. If there exists a $y \in K \setminus \bigcup_{j=1}^{\infty} Q_{x_j}$, this y would belong to all K_j, $j = 1,2,\dots$, and then $s_j \ge \ell(Q_y)$ for all j. Since some subsequence of the s_j's tends to zero, it would follow that $\ell(Q_y) = 0$, which would force the open cube Q_y to be the empty set, a contradiction. Thus (7.1.31) holds.

Finally, we show that $\sum_{j=1}^{m} \chi_{Q_{x_j}}(y) \le 72^n$ for almost every point $y \in \mathbf{R}^n$. To prove this we consider the n hyperplanes H_i that are parallel to the coordinate hyperplanes and pass through the point y. Then we write $\mathbf{R}^n$ as a union of ~~n hyperplanes H_i of n-dimensional Lebesgue measure zero and~~ 2^n higher-dimensional ~~open~~ closed "octants" O_r, henceforth called orthants, determined by the H_i's. We fix a $y \in \mathbf{R}^n$ and we show that there are at most 36^n points $x_j \in O_r$ such that y lies in Q_{x_j} for a given open orthant O_r. To prove this assertion, setting $|z|_{\ell^\infty} = \sup_{1\le i\le n}|z_i|$ for points $z = (z_1,\dots,z_n)$ in $\mathbf{R}^n$, we pick an $x_{k_0} \in K \cap O_r$ such that $Q_{x_{k_0}}$ contains y and $|x_{k_0} - y|_{\ell^\infty}$ is the largest possible among all $|x_j - y|_{\ell^\infty}$. If x_j is another point in $K \cap O_r$ such that Q_{x_j} contains y, then we claim that $x_j \in Q_{x_{k_0}}$. Indeed, to show this we notice that for each $i \in \{1,\dots,n\}$ we have

$$\begin{aligned} |x_{j,i} - x_{k_0,i}| &= \big|x_{j,i} - y_i - (x_{k_0,i} - y_i)\big| \\ &= \big||x_{j,i} - y_i| - |x_{k_0,i} - y_i|\big| \end{aligned}$$

$$\begin{aligned}
&\leq \max\left(|x_{k_0,i}-y_i|,|x_{j,i}-y_i|\right)\\
&\leq \max\left(|x_{k_0}-y|_{\ell^\infty},|x_j-y|_{\ell^\infty}\right)\\
&= |x_{k_0}-y|_{\ell^\infty}\\
&< \tfrac{1}{2}\ell(Q_{x_{k_0}}),
\end{aligned}$$

where the second equality is due to the fact that x_j, x_{k_0} lie in the same orthant and the last inequality in the fact that $y \in Q_{x_{k_0}}$; it follows that x_j lies in $Q_{x_{k_0}}$.

We observed previously that $i > j$ implies $x_i \notin Q_{x_j}$. Since x_j lies in $Q_{x_{k_0}}$, one must then have $j \leq k_0$, which implies that $\frac{1}{2}\ell(Q_{x_{k_0}}) < \ell(Q_{x_j})$. Thus all cubes Q_{x_j} with centers in $K \cap O_r$ that contain the fixed point y have side lengths comparable to that of $Q_{x_{k_0}}$. A simple geometric argument now gives that there are at most finitely many cubes Q_{x_j} of side length between α and 6α that contain the given point y such that $\frac{1}{3}Q_{x_j}$ are pairwise disjoint. Indeed, let $\alpha = \frac{1}{2}\ell(Q_{x_{k_0}})$ and let $\{Q_{x_r}\}_{r\in I}$ be the cubes with these properties. Then we have

$$\frac{\alpha^n|I|}{3^n} \leq \sum_{r\in I}\left|\tfrac{1}{3}Q_{x_r}\right| = \Big|\bigcup_{r\in I}\tfrac{1}{3}Q_{x_r}\Big| \leq \Big|\bigcup_{r\in I}Q_{x_r}\Big| \leq (12\alpha)^n,$$

since all the cubes Q_{x_r} contain the point y and have length at most 6α and they must therefore be contained in a cube of side length 12α centered at y. This observation shows that $|I| \leq 36^n$, and since there are 2^n sets O_r, we conclude the proof of (7.1.32). □

Remark 7.1.11. Without use of the covering Lemma 7.1.10, (7.1.29) can be proved via the doubling property of w (cf. Exercise 2.1.1(a)), but then the resulting constant $C(q,n)$ would depend on the doubling constant of the measure $w\,dx$ and thus on $[w]_{A_p}$; this would yield a worse dependence on $[w]_{A_p}$ in the constant in (7.1.25).

Exercises

7.1.1. Let k be a nonnegative measurable function such that k, k^{-1} are in $L^\infty(\mathbf{R}^n)$. Prove that if w is an A_p weight for some $1 \leq p < \infty$, then so is kw.

7.1.2. Let w_1, w_2 be two A_1 weights and let $1 < p < \infty$. Prove that $w_1 w_2^{1-p}$ is an A_p weight by showing that

$$[w_1 w_2^{1-p}]_{A_p} \leq [w_1]_{A_1}[w_2]_{A_1}^{p-1}.$$

7.1.3. Suppose that $w \in A_p$ for some $p \in [1,\infty)$ and $0 < \delta < 1$. Prove that $w^\delta \in A_q$, where $q = \delta p + 1 - \delta$, by showing that

$$[w^\delta]_{A_q} \leq [w]_{A_p}^{\delta}.$$