Example 7.1.8. On $\mathbf{R}^n$ the function

$$u(x) = \begin{cases} \log \frac{1}{|x|} & \text{when } |x| < \frac{1}{e}, \\ 1 & \text{otherwise,} \end{cases}$$

is an A_1 weight. Indeed, to check condition (7.1.19) it suffices to consider balls of type I and type II as defined in Example 7.1.6. In either case the required estimate follows easily.

We now return to a point alluded to earlier, that the A_p condition implies the boundedness of the Hardy–Littlewood maximal function M on the space $L^p(w)$. To this end we introduce four maximal functions acting on functions f that are locally integrable with respect to w:

$$M^w(f)(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f|\, w\, dy,$$

where the supremum is taken over open balls B that contain the point x and

$$\begin{aligned} \mathcal{M}^w(f)(x) &= \sup_{\delta>0} \frac{1}{w(B(x,\delta))} \int_{B(x,\delta)} |f|\, w\, dy, \\ M_c^w(f)(x) &= \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f|\, w\, dy, \end{aligned}$$

where Q is an open cube containing the point x, and

$$\mathcal{M}_c^w(f)(x) = \sup_{\delta>0} \frac{1}{w(Q(x,\delta))} \int_{Q(x,\delta)} |f|\, w\, dy,$$

where $Q(x,\delta) = \prod_{j=1}^n (x_j - \delta, x_j + \delta)$ is a cube of side length 2δ centered at $x = (x_1, \dots, x_n)$. When $w = 1$, these maximal functions reduce to the standard ones $M(f)$, $\mathcal{M}(f)$, $M_c(f)$, and $\mathcal{M}_c(f)$, the uncentered and centered Hardy–Littlewood maximal functions with respect to balls and cubes, respectively.

Theorem 7.1.9. *(a) Let $w \in A_1$. Then we have*

$$\big\|\mathcal{M}_c\big\|_{L^1(w) \to L^{1,\infty}(w)} \le 3^n [w]_{A_1}^2 . \tag{7.1.24}$$

(b) Let $w \in A_p(\mathbf{R}^n)$ for some $1 < p < \infty$. Then there is a constant $C_{n,p}$ such that

$$\big\|\mathcal{M}_c\big\|_{L^p(w) \to L^p(w)} \le C_{n,p} [w]_{A_p}^{\frac{1}{p-1}} . \tag{7.1.25}$$

Since the operators $\mathcal{M}_c$, M_c, $\mathcal{M}$, and M are pointwise comparable, a similar conclusions hold for the other three as well.

Proof. (a) Since $d\mu = w dx$ is a doubling measure and $d\mu(3Q) \le 3^n [w]_{A_1} \mu(Q)$, using Proposition 7.1.5 (9) and Exercise 2.1.1 we obtain that M_c^w maps $L^1(w)$ to $L^{1,\infty}(w)$ with norm at most $3^n [w]_{A_1}$. This proves (7.1.24) since $M_c \le [w]_{A_1} M_c^w$.