is called the A_1 *Muckenhoupt characteristic constant* of w, or simply the A_1 *characteristic constant* of w. Note that A_1 weights w satisfy

$$\frac{1}{|Q|}\int_Q w(t)\,dt \le [w]_{A_1} \operatorname*{ess.inf}_{y\in Q} w(y) \tag{7.1.18}$$

for all cubes Q in $\mathbf{R}^n$.

Remark 7.1.2. We also define

$$[w]_{A_1}^{\text{balls}} = \sup_{B \text{ balls in } \mathbf{R}^n} \left(\frac{1}{|B|}\int_B w(t)\,dt\right)\big\|w^{-1}\big\|_{L^\infty(B)}. \tag{7.1.19}$$

Using (7.1.13), we see that the smallest constant C_1 that appears in (7.1.16) is equal to the A_1 characteristic constant of w as defined in (7.1.19). This is also equal to the smallest constant that appears in (7.1.13). All these constants are bounded above and below by dimensional multiples of $[w]_{A_1}$.

We now recall condition (7.1.5), which motivates the following definition of A_p weights for $1 < p < \infty$.

Definition 7.1.3. Let $1 < p < \infty$. A weight w is said to be *of class* A_p if

$$\sup_{Q \text{ cubes in } \mathbf{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty. \tag{7.1.20}$$

The expression in (7.1.20) is called the A_p *Muckenhoupt characteristic constant of* w (or simply the A_p characteristic constant of w) and is denoted by $[w]_{A_p}$.

Remark 7.1.4. Note that Definitions 7.1.1 and 7.1.3 could have been given with the set of all cubes in $\mathbf{R}^n$ replaced by the set of all balls in $\mathbf{R}^n$. Defining $[w]_{A_p}^{\text{balls}}$ as in (7.1.20) except that cubes are replaced by balls, we see that

$$\big(2^n/v_n\big)^p \le \frac{[w]_{A_p}}{[w]_{A_p}^{\text{balls}}} \le \big(n^{n/2}v_n 2^{-n}\big)^p. \tag{7.1.21}$$

7.1.2 *Properties of* A_p *Weights*

It is straightforward that translations, isotropic dilations, and scalar multiples of A_p weights are also A_p weights with the same A_p characteristic. We summarize some basic properties of A_p weights in the following proposition.

Proposition 7.1.5. *Let $w \in A_p$ for some $1 \le p < \infty$. Then*

(1) $[\delta^\lambda(w)]_{A_p} = [w]_{A_p}$, *where* $\delta^\lambda(w)(x) = w(\lambda x_1,\dots,\lambda x_n)$.

(2) $[\tau^z(w)]_{A_p} = [w]_{A_p}$, *where* $\tau^z(w)(x) = w(x-z)$, $z \in \mathbf{R}^n$.

(3) $[\lambda w]_{A_p} = [w]_{A_p}$ *for all* $\lambda > 0$.

(4) When $1 < p < \infty$*, the function* $w^{-\frac{1}{p-1}}$ *is in* $A_{p'}$ *with characteristic constant*

$$\big[w^{-\frac{1}{p-1}}\big]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$

Therefore, $w \in A_2$ *if and only if* $w^{-1} \in A_2$ *and both weights have the same* A_2 *characteristic constant.*

(5) $[w]_{A_p} \geq 1$ *for all* $w \in A_p$*. Equality holds if and only if w is a constant.*

(6) The classes A_p *are increasing as p increases; precisely, for* $1 \leq p < q < \infty$ *we have*

$$[w]_{A_q} \leq [w]_{A_p}.$$

(7) $\lim_{q \to 1+} [w]_{A_q} = [w]_{A_1}$ *if* $w \in A_1$.

(8) When $p > 1$*, the following is an equivalent characterization of the* A_p *characteristic constant of w:*

$$[w]_{A_p} = \sup_{\substack{Q \, cubes \\ in \, \mathbf{R}^n}} \sup_{\substack{f \in L^p(Q, w\,dt) \\ \int_Q |f|^p w\,dt > 0}} \left\{ \frac{\left(\frac{1}{|Q|}\int_Q |f(t)|\,dt\right)^p}{\frac{1}{w(Q)}\int_Q |f(t)|^p w(t)\,dt} \right\}.$$

(9) The measure $w(x)\,dx$ *is doubling: precisely, for all* $\lambda > 1$ *and all cubes Q we have*

$$w(\lambda Q) \leq \lambda^{np}[w]_{A_p}\, w(Q).$$

(λQ *denotes the cube with the same center as Q and side length* λ *times the side length of Q.)*

Proof. The simple proofs of (1), (2), and (3) are left as an exercise. Property (4) is also easy to check and plays the role of duality in this context. To prove (5) we use Hölder's inequality with exponents p and p' to obtain

$$1 = \frac{1}{|Q|}\int_Q dx = \frac{1}{|Q|}\int_Q w(x)^{\frac{1}{p}} w(x)^{-\frac{1}{p}}\,dx \leq [w]_{A_p}^{\frac{1}{p}},$$

with equality holding only when $w(x)^{\frac{1}{p}} = c\,w(x)^{-\frac{1}{p}}$ for some $c > 0$ (i.e., when w is a constant). To prove (6), observe that $0 < q' - 1 < p' - 1 \leq \infty$ and that the statement

$$[w]_{A_q} \leq [w]_{A_p}$$

is equivalent to the fact

$$\big\|w^{-1}\big\|_{L^{q'-1}(Q, \frac{dx}{|Q|})} \leq \big\|w^{-1}\big\|_{L^{p'-1}(Q, \frac{dx}{|Q|})}.$$

Property (7) is a consequence of part (a) of Exercise 1.1.3.

To prove (8), apply Hölder's inequality with exponents p and p' to get

$$\begin{aligned}
(\operatorname*{Avg}_Q |f|)^p &= \left(\frac{1}{|Q|}\int_Q |f(x)|\,dx\right)^p \\
&= \left(\frac{1}{|Q|}\int_Q |f(x)|w(x)^{\frac{1}{p}}w(x)^{-\frac{1}{p}}\,dx\right)^p \\
&\le \frac{1}{|Q|^p}\left(\int_Q |f(x)|^p w(x)\,dx\right)\left(\int_Q w(x)^{-\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}} \\
&= \left(\frac{1}{w(Q)}\int_Q |f(x)|^p w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \\
&\le [w]_{A_p}\left(\frac{1}{w(Q)}\int_Q |f(x)|^p w(x)\,dx\right).
\end{aligned}$$

This argument proves the inequality $\geq$ in (8) when $p > 1$. ~~In the case $p = 1$ the obvious modification yields the same inequality.~~ The reverse inequality follows by taking $f = (w+\varepsilon)^{-p'/p}$ as in (7.1.6) and letting $\varepsilon \to 0$.

Applying (8) to the function $f = \chi_Q$ and putting λQ in the place of Q in (8), we obtain

$$w(\lambda Q) \le \lambda^{np}[w]_{A_p}w(Q),$$

which says that $w(x)\,dx$ is a doubling measure. This proves (9). □

Example 7.1.6. A positive measure $d\mu$ is called doubling if for some $C < \infty$,

$$\mu(2B) \le C\mu(B) \tag{7.1.22}$$

for all balls B. We show that the measures $|x|^a\,dx$ are doubling when $a > -n$. We divide all balls $B(x_0,R)$ in $\mathbf{R}^n$ into two categories: balls of type I that satisfy $|x_0| \ge 3R$ and type II that satisfy $|x_0| < 3R$. For balls of type I we observe that

$$\int_{B(x_0,2R)} |x|^a\,dx \le v_n(2R)^n \begin{cases} (|x_0|+2R)^a & \text{when } a \ge 0, \\ (|x_0|-2R)^a & \text{when } a < 0, \end{cases}$$

$$\int_{B(x_0,R)} |x|^a\,dx \ge v_nR^n \begin{cases} (|x_0|-R)^a & \text{when } a \ge 0, \\ (|x_0|+R)^a & \text{when } a < 0. \end{cases}$$

Since $|x_0| \ge 3R$, we have $|x_0|+2R \le 4(|x_0|-R)$ and $|x_0|-2R \ge \frac{1}{4}(|x_0|+R)$, from which (7.1.22) follows with $C = 2^{3n}4^{|a|}$.

For balls of type II, we have $|x_0| \le 3R$ and we note two things: first

$$\int_{B(x_0,2R)} |x|^a\,dx \le \int_{|x|\le 5R} |x|^a\,dx = c_nR^{n+a},$$