It is an interesting observation that such functions are completely determined by their values at the points $x = k/2B$, where $k \in \mathbf{Z}^n$. We have the following result.

Theorem 6.6.9. *(a) Let f in $L^1(\mathbf{R}^n)$ be band limited on the cube $[-B,B]^n$. Then f can be sampled by its values at the points $x = k/2B$, where $k \in \mathbf{Z}^n$. In particular, we have*

$$f(x_1,\dots,x_n) = \sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B}\Big)\prod_{j=1}^{n}\frac{\sin(2\pi Bx_j - \pi k_j)}{2\pi Bx_j - \pi k_j} \tag{6.6.18}$$

for almost all $x \in \mathbf{R}^n$.
(b) Suppose that $f \in L^1(\mathbf{R}^n)$ is band-limited on the cube $[-B',B']^n$ where $0 < B' < B$. Then f can be sampled by its values at the points $x = k/2B$, $k \in \mathbf{Z}^n$ as follows

$$f(x_1,\dots,x_n) = \sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B}\Big)\Phi(x-k)\,, \tag{6.6.19}$$

for some Schwartz function Φ that depends on B,B'.

Proof. Since the function $\widehat{f}$ is supported in $[-B,B]^n$, we use Exercise 6.6.2 to obtain

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{(2B)^n}\sum_{k\in\mathbf{Z}^n}\widehat{\widehat{f}}\Big(\frac{k}{2B}\Big)e^{2\pi i\frac{k}{2B}\cdot\xi}\\ &= \frac{1}{(2B)^n}\sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big)e^{2\pi i\frac{k}{2B}\cdot\xi}\,.\end{aligned}$$

Inserting this identity in the inversion formula

$$f(x) = \int_{[-B,B]^n}\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi\,,$$

which holds for almost all $x \in \mathbf{R}^n$ since $\widehat{f}$ is continuous and therefore integrable over $[-B,B]^n$, we obtain

$$\begin{aligned}f(x) &= \int_{[-B,B]^n}\frac{1}{(2B)^n}\sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big)e^{2\pi i\frac{k}{2B}\cdot\xi}e^{2\pi i x\cdot\xi}\,d\xi\\ &= \sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big)\frac{1}{(2B)^n}\int_{[-B,B]^n}e^{2\pi i(\frac{k}{2B}+x)\cdot\xi}\,d\xi \qquad (6.6.20)\\ &= \sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big)\prod_{j=1}^{n}\frac{\sin(2\pi Bx_j+\pi k_j)}{2\pi Bx_j+\pi k_j}\,. \qquad (6.6.21)\end{aligned}$$

This is exactly (6.6.18) when we change k to $-k$ and thus part (a) is proved. For part (b) we argue similarly, except that we replace $\chi_{[-B,B]^n}$ by $\widehat{\Phi}$, where $\widehat{\Phi}$ is smooth, equal to 1 on $[-B',B']^n$ and vanishes outside $[-B,B]^n$. Then we can insert the function $\widehat{\Phi}(\xi)$ in (6.6.20) and instead of (6.6.21) we obtain the expression on the right in (6.6.19). $\square$

Remark 6.6.10. Identity (6.6.18) holds for any $B'' > B$. In particular, we have

$$\sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B}\Big)\prod_{j=1}^{n}\frac{\sin(2\pi Bx_j-\pi k_j)}{2\pi Bx_j-\pi k_j} = \sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B''}\Big)\prod_{j=1}^{n}\frac{\sin(2\pi B''x_j-\pi k_j)}{2\pi B''x_j-\pi k_j}$$

for all $x\in\mathbf{R}^n$ whenever f is band-limited in $[-B,B]^n$. In particular, band-limited functions in $[-B,B]^n$ can be sampled by their values at the points $k/2B''$ for any $B''\geq B$.

However, band-limited functions in $[-B,B]^n$ cannot be sampled by the points $k/2B'$ for any $B'<B$, as the following example indicates.

Example 6.6.11. For $0<B'<B$, let $f(x)=g(x)\sin(2\pi B'x)$, where $\widehat{g}$ is supported in the interval $[-(B-B'),B-B']$. Then f is band limited in $[-B,B]$, but it cannot be sampled by its values at the points $k/2B'$, since it vanishes at these points and f is not identically zero if g is not the zero function.

Next, we give a couple of results that relate the L^p norm of a given function with the ℓ^p norm (or quasi-norm) of its sampled values.

Theorem 6.6.12. *Let f be a tempered*[2] *function whose Fourier transform is supported in the closed ball $\overline{B(0,t)}$ for some $0<t<\infty$. Assume that f lies in $L^p(\mathbf{R}^n)$ for some $0<p\leq\infty$. Then there is a constant $C(n,p)$ such that*

$$\big\|\{f(k)\}_{k\in\mathbf{Z}^n}\big\|_{\ell^p(\mathbf{Z}^n)}\leq C(n,p)\,(1+t)\,(1+t^{\frac{2n}{p}})\big\|f\big\|_{L^p(\mathbf{R}^n)}.$$

Proof. The proof is based on the following fact, whose proof can be found in [131] (Lemma 2.2.3). Let $0<r<\infty$. Then there exists a constant $C_2=C_2(n,r)$ such that for all $t>0$ and for all $\mathscr{C}^1$ functions u on $\mathbf{R}^n$ whose distributional Fourier transform is supported in the ball $|\xi|\leq t$ we have

$$\sup_{z\in\mathbf{R}^n}\frac{1}{t}\,\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\leq C_2\,M(|u|^r)(x)^{\frac{1}{r}}\,, \tag{6.6.22}$$

where M denotes the Hardy–Littlewood maximal operator.

Notice that f is a $\mathscr{C}^\infty$ function since its Fourier transform is compactly supported. Assuming (6.6.22), for each $k\in\mathbf{Z}^n$ and $x\in[0,1]^n$ we use the mean value theorem to obtain

$$\begin{aligned}|f(k)| &\leq |f(x+k)|+\sqrt{n}\sup_{z\in[0,1]^n}|\nabla f(z+k)|\\ &\leq |f(x+k)|+\sqrt{n}\sup_{z\in B(x+k,\sqrt{n})}|\nabla f(z)|\,.\end{aligned}$$

We raise this inequality to the power p, we integrate over the cube $[0,1]^n$, we sum over $k\in\mathbf{Z}^n$, and then we take the $1/p$ power. Let $c_p=\max(1,2^{1/p-1})$ and $c(n,r,t)=$

[2] A function is called tempered if there are constants C,M such that $|f(x)|\leq C(1+|x|)^M$ for all $x\in\mathbf{R}^n$. Tempered functions are tempered distributions.

$\sqrt{n}\,t(1+t\sqrt{n})^{n/r}$. The sum over k and the integral over $[0,1]^n$ yield an integral over $\mathbf{R}^n$ and thus we obtain

$$\begin{aligned}
\Big[\sum_{k\in\mathbf{Z}^n}|f(k)|^p\Big]^{\frac{1}{p}} &\le \Big[\int_{\mathbf{R}^n}|f(x)+\sqrt{n}\sup_{z\in B(x,\sqrt{n})}|\nabla f(z)|^p\,dx\Big]^{\frac{1}{p}}\\
&\le c_p\Big[\big\|f\big\|_{L^p}+\sqrt{n}\Big(\int_{\mathbf{R}^n}\sup_{z\in B(0,\sqrt{n})}|\nabla f(x-z)|^p\,dx\Big)^{\frac{1}{p}}\Big]\\
&\le c_p\Big[\big\|f\big\|_{L^p}+c(n,r,t)\Big(\int_{\mathbf{R}^n}\Big\{\sup_{z\in B(0,\sqrt{n})}\frac{|\nabla f(x-z)|}{t(1+t|z|)^{\frac{n}{r}}}\Big\}^p dx\Big)^{\frac{1}{p}}\Big]\\
&\le c_p\Big[\big\|f\big\|_{L^p}+c(n,r,t)\Big(\int_{\mathbf{R}^n}\Big\{\sup_{z\in\mathbf{R}^n}\frac{|\nabla f(x-z)|}{t(1+t|z|)^{\frac{n}{r}}}\Big\}^p dx\Big)^{\frac{1}{p}}\Big]\\
&\le c_p\Big[\big\|f\big\|_{L^p}+c(n,r,t)C_2\Big(\int_{\mathbf{R}^n}[M(|f|^r)(x)]^{\frac{p}{r}}\,dx\Big)^{\frac{1}{p}}\Big],
\end{aligned}$$

where the last step uses (6.6.22). We now select $r=p/2$ if $p<\infty$ and r to be any number if $p=\infty$. The required inequality follows from the boundedness of the Hardy-Littlewood maximal operator on L^2 if $p<\infty$ or on L^∞ if $p=\infty$. □

The next theorem could be considered a partial converse of Theorem 6.6.12.

Theorem 6.6.13. *Suppose that an integrable function f has Fourier transform supported in the cube $[-(\frac{1}{2}-\varepsilon),\frac{1}{2}-\varepsilon]^n$ for some $0<\varepsilon<1/2$. Furthermore, suppose that the sequence of coefficients $\{f(k)\}_{k\in\mathbf{Z}^n}$ lies in $\ell^p(\mathbf{Z}^n)$ for some $0<p\le\infty$. Then f lies in $L^p(\mathbf{R}^n)$ and the following estimate is valid*

$$\big\|f\big\|_{L^p(\mathbf{R}^n)}\le C_{n,p,\varepsilon}\big\|\{f(k)\}_k\big\|_{\ell^p(\mathbf{Z}^n)}. \tag{6.6.23}$$

Proof. We fix a smooth function $\widehat{\Phi}$ supported in $[-\frac{1}{2},\frac{1}{2}]^n$ and equal to 1 on the smaller cube $[-(\frac{1}{2}-\varepsilon),\frac{1}{2}-\varepsilon]^n$. Then we may write $f=f*\Phi$, since $\widehat{\Phi}$ is equal to one on the support of $\widehat{f}$. Writing $\widehat{f}$ in terms of its Fourier series we have

$$\widehat{f}(\xi)=\sum_{k\in\mathbf{Z}^n}\widehat{\widehat{f}}(k)e^{2\pi ik\cdot\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]^n}=\sum_{k\in\mathbf{Z}^n}f(-k)e^{2\pi ik\cdot\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]^n} \tag{6.6.24}$$

Since f is integrable, $\widehat{f}$ is continuous and thus integrable over $[-\frac{1}{2},\frac{1}{2}]^n$. By Fourier inversion we have

$$f(x)=\int_{[-\frac{1}{2},\frac{1}{2}]^n}\widehat{f}(\xi)e^{2\pi ix\cdot\xi}d\xi=\int_{[-\frac{1}{2},\frac{1}{2}]^n}\widehat{f}(\xi)\widehat{\Phi}(\xi)e^{2\pi ix\cdot\xi}d\xi \tag{6.6.25}$$

for almost all $x\in\mathbf{R}^n$. Inserting (6.6.25) in (6.6.24) we obtain

$$f(x)=\int_{[-\frac{1}{2},\frac{1}{2}]^n}\sum_{k\in\mathbf{Z}^n}f(-k)e^{2\pi ik\cdot\xi}e^{2\pi ix\cdot\xi}\widehat{\Phi}(\xi)\,d\xi$$