which is easily shown to be in $L^1(\mathbf{T}^n)$. Moreover, we have

$$\widehat{G}(m) = \widehat{g}(m)$$

for all $m \in \mathbf{Z}^n$, where $\widehat{G}(m)$ denotes the mth Fourier coefficient of G and $\widehat{g}(m)$ denotes the Fourier transform of g at $\xi = m$. If $\widehat{g}(m) = 0$ for all $m \in \mathbf{Z}^n \setminus \{0\}$, then all the Fourier coefficients of G (except for $m = 0$) vanish, which means that the sequence $\{\widehat{G}(m)\}_{m \in \mathbf{Z}^n}$ lies in $\ell^1(\mathbf{Z}^n)$ and hence Fourier inversion applies. We conclude that for almost all $x \in \mathbf{T}^n$ we have

$$G(x) = \sum_{m \in \mathbf{Z}^n} \widehat{G}(m) e^{2\pi i m \cdot x} = \widehat{G}(0) = \widehat{g}(0) = \int_{\mathbf{R}^n} g(t)\, dt\,.$$

Conversely, if G is a constant, then $\widehat{G}(m) = 0$ for all $m \in \mathbf{Z}^n \setminus \{0\}$, and so the same holds for g. □

A consequence of the preceding proposition is the following.

Proposition 6.6.4. *Let $\varphi \in L^2(\mathbf{R}^n)$. Then the sequence*

$$\{\varphi(x-k)\}_{k \in \mathbf{Z}^n} \tag{6.6.3}$$

forms an orthonormal set in $L^2(\mathbf{R}^n)$ if and only if

$$\sum_{k \in \mathbf{Z}^n} |\widehat{\varphi}(\xi + k)|^2 = 1 \tag{6.6.4}$$

for almost all $\xi \in \mathbf{R}^n$.

Proof. Observe that either (6.6.4) or the hypothesis that the sequence in (6.6.3) is orthonormal implies that $\|\varphi\|_{L^2} = 1$. Also the orthonormality condition

$$\int_{\mathbf{R}^n} \varphi(x-k)\overline{\varphi(x-j)}\, dx = \begin{cases} 1 & \text{when } j = k, \\ 0 & \text{when } j \neq k, \end{cases}$$

is equivalent to

$$\int_{\mathbf{R}^n} e^{-2\pi i k \cdot \xi} \widehat{\varphi}(\xi) \overline{e^{-2\pi i j \cdot \xi} \widehat{\varphi}(\xi)}\, d\xi = (|\widehat{\varphi}|^2)\widehat{\ }(k-j) = \begin{cases} 1 & \text{when } j = k, \\ 0 & \text{when } j \neq k, \end{cases}$$

in view of Parseval's identity. Proposition 6.6.3 with $g(\xi) = |\widehat{\varphi}(\xi)|^2$ gives that the latter is equivalent to

$$\sum_{k \in \mathbf{Z}^n} |\widehat{\varphi}(\xi + k)|^2 = \int_{\mathbf{R}^n} |\widehat{\varphi}(t)|^2\, dt = 1$$

for almost all $\xi \in \mathbf{R}^n$. □

Corollary 6.6.5. *Let $\varphi \in L^2(\mathbf{R}^n)$ and suppose that the sequence*

$$\{\varphi(x-k)\}_{k\in\mathbf{Z}^n} \tag{6.6.5}$$

forms an orthonormal set in $L^2(\mathbf{R}^n)$. Then the measure of the support of $\widehat{\varphi}$ is at least 1, that is,

$$|\mathrm{supp}\,\widehat{\varphi}| \geq 1\,. \tag{6.6.6}$$

Moreover, if $|\mathrm{supp}\,\widehat{\varphi}| = 1$, then $|\widehat{\varphi}(\xi)| = 1$ for almost all $\xi \in \mathrm{supp}\,\widehat{\varphi}$.

Proof. It follows from (6.6.4) that $|\widehat{\varphi}| \leq 1$ for almost all $\xi \in \mathbf{R}^n$ and thus

$$|\mathrm{supp}\,\widehat{\varphi}| \geq \int_{\mathbf{R}^n} |\widehat{\varphi}(\xi)|^2\,d\xi = \int_{[0,1)^n} \sum_{k\in\mathbf{Z}^n} |\widehat{\varphi}(\xi+k)|^2\,d\xi = \int_{[0,1)^n} 1\,d\xi = 1\,.$$

If equality holds in (6.6.6), then equality holds in the preceding inequality, and since $|\widehat{\varphi}| \leq 1$ a.e., it follows that $|\widehat{\varphi}(\xi)| = 1$ for almost all ξ in $\mathrm{supp}\,\widehat{\varphi}$. □

6.6.2 Construction of a Nonsmooth Wavelet

Having established these preliminary facts, we now start searching for examples of wavelets. It follows from Corollary 6.6.5 that the support of the Fourier transform of a wavelet must have measure at least 1. It is reasonable to ask whether this support can have measure exactly 1. Example 6.6.6 indicates that this can indeed happen. As dictated by the same corollary, the Fourier transform of such a wavelet must satisfy $|\widehat{\varphi}(\xi)| = 1$ for almost all $\xi \in \mathrm{supp}\,\widehat{\varphi}$, so it is natural to look for a wavelet φ such that $\widehat{\varphi} = \chi_A$ for some set A. We can start by asking whether the function

$$\widehat{\varphi} = \chi_{[-\frac{1}{2},\frac{1}{2}]}$$

on $\mathbf{R}$ is an appropriate Fourier transform of a wavelet, but a moment's thought shows that the functions $\varphi_{\mu,0}$ and $\varphi_{\nu,0}$ cannot be orthogonal to each other when $\mu \neq 0$. The problem here is that the Fourier transforms of the functions $\varphi_{\nu,k}$ cluster near the origin and do not allow for the needed orthogonality. We can fix this problem by considering a function whose Fourier transform vanishes near the origin. Among such functions, a natural candidate is

$$\chi_{[-1,-\frac{1}{2})} + \chi_{(\frac{1}{2},1]}\,, \tag{6.6.7}$$

which is indeed the Fourier transform of a wavelet.

Example 6.6.6. Let $A = [-1,-\frac{1}{2}) \bigcup (\frac{1}{2},1]$ and define a function φ on $\mathbf{R}$ by setting

$$\widehat{\varphi} = \chi_A\,.$$

Then we assert that the family of functions

$$\{\varphi_{\nu,k}(x)\}_{k\in\mathbf{Z},\nu\in\mathbf{Z}} = \{2^{\nu/2}\varphi(2^\nu x-k)\}_{k\in\mathbf{Z},\nu\in\mathbf{Z}}$$

is an orthonormal basis of $L^2(\mathbf{R})$ (i.e., the function φ is a wavelet). This is an example of a wavelet with *minimally supported frequency*.

To verify this assertion, first note that $\{\varphi_{0,k}\}_{k\in\mathbf{Z}}$ is an orthonormal set, since (6.6.4) is easily seen to hold. Dilating by 2^ν, it follows that $\{\varphi_{\nu,k}\}_{k\in\mathbf{Z}}$ is also an orthonormal set for every fixed $\nu\in\mathbf{Z}$. Next, observe that if $\mu\neq\nu$, then

$$\operatorname{supp}\widehat{\varphi_{\nu,k}}\cap\operatorname{supp}\widehat{\varphi_{\mu,l}}=\emptyset. \tag{6.6.8}$$

This implies that the family $\{2^{\nu/2}\varphi(2^\nu x-k)\}_{k\in\mathbf{Z},\nu\in\mathbf{Z}}$ is also orthonormal.

Next, we observe that the completeness of $\{\varphi_{\nu,k}\}_{\nu,k\in\mathbf{Z}}$ is equivalent to that of $\{\widehat{\varphi_{\nu,k}}(\xi)\}_{\nu,k\in\mathbf{Z}} = \{2^{-\nu/2}e^{-2\pi ik\xi 2^{-\nu}}\chi_{2^\nu A}(\xi)\}_{\nu,k\in\mathbf{Z}}$. Let $f\in L^2(\mathbf{R})$, fix any $\nu\in\mathbf{Z}$, and define

$$h(\xi)=2^{\nu/2}f(2^\nu\xi).$$

Suppose that for all $k\in\mathbf{Z}$,

$$\begin{aligned}0=\langle f,\widehat{\varphi_{\nu,k}}\rangle &= \int_{2^\nu A} f(\xi)2^{-\nu/2}e^{-2\pi ik\xi 2^{-\nu}}\,d\xi\\ &= \int_A 2^{\nu/2}f(2^\nu\xi)e^{-2\pi ik\xi}\,d\xi\\ &= \langle\chi_A h, e^{-2\pi ik\xi}\rangle.\end{aligned}$$

Exercise 6.6.1(a) shows $\{e^{-2\pi ik\xi}\}_{k\in\mathbf{Z}}$ is an orthonormal basis of $L^2(A)$, and therefore $\chi_A h=0$ almost everywhere. From the definition of h it follows that $\chi_{2^\nu A}f=0$ almost everywhere. Now suppose for all $\nu,k\in\mathbf{Z}$

$$0=\langle f,\widehat{\varphi_{\nu,k}}\rangle.$$

Then $\chi_{2^\nu A}f=0$ almost everywhere for all $\nu\in\mathbf{Z}$. Since $\cup_{\nu\in\mathbf{Z}}2^\nu A=\mathbf{R}\setminus\{0\}$, it follows that $f=0$ almost everywhere. We conclude $\{\widehat{\varphi_{\nu,k}}\}_{\nu,k\in\mathbf{Z}}$ is complete.

6.6.3 Construction of a Smooth Wavelet

The wavelet basis of $L^2(\mathbf{R})$ constructed in Example 6.6.6 is forced to have slow decay at infinity, since the Fourier transforms of the elements of the basis are nonsmooth. Smoothing out the function $\widehat{\varphi}$ but still expecting φ to be wavelet is a bit tricky, since property (6.6.8) may be violated when $\mu\neq\nu$, and moreover, (6.6.4) may be destroyed. These two obstacles are overcome by the careful construction of the next theorem.