Rephrasing the question posed earlier, the main issue addressed in this section is whether smooth wavelets actually exist. Before we embark on this topic, we recall that we have already encountered examples of nonsmooth wavelets.

Example 6.6.2. (The Haar wavelet) Recall the family of functions

$$h_I(x) = |I|^{-\frac{1}{2}}(\chi_{I_L} - \chi_{I_R}),$$

where I ranges over $\mathscr{D}$ (the set of all dyadic intervals) and I_L is the left part of I and I_R is the right part of I. Note that if $I = [2^{-\nu}k, 2^{-\nu}(k+1))$, then

$$h_I(x) = 2^{\frac{\nu}{2}}\varphi(2^\nu x - k),$$

where $x \in \mathbf{R}$ and

$$\varphi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}. \tag{6.6.2}$$

The single function φ in (6.6.2) therefore generates the Haar basis by taking translations and dilations. Moreover, we observed in Section 6.4 that the family $\{h_I\}_I$ is orthonormal. Moreover, in Theorem 6.4.6 we obtained the representation

$$f = \sum_{I\in\mathscr{D}} \langle f, h_I\rangle h_I \qquad \text{in } L^2,$$

which proves the completeness of the system $\{h_I\}_{I\in\mathscr{D}}$ in $L^2(\mathbf{R})$.

6.6.1 Some Preliminary Facts

Before we look at more examples, we make some observations. We begin with the following useful fact.

Proposition 6.6.3. *Let $g \in L^1(\mathbf{R}^n)$. Then*

$$\widehat{g}(m) = 0 \qquad \text{for all } m \in \mathbf{Z}^n \setminus \{0\}$$

if and only if

$$\sum_{k\in\mathbf{Z}^n} g(x+k) = \int_{\mathbf{R}^n} g(t)\,dt$$

for almost all $x \in \mathbf{T}^n$.

Proof. We define the periodic function

$$G(x) = \sum_{k\in\mathbf{Z}^n} g(x+k),$$