6.5.4 Completion of the Proof

It remains to combine the previous ingredients to complete the proof of the theorem. Interpolating between the $L^2 \to L^2$ and $L^1 \to L^{1,\infty}$ estimates obtained in Lemmas 6.5.2 and 6.5.3, we obtain

$$\big\|\mathscr{M}_j(f)\big\|_{L^p(\mathbf{R}^n)} \le C_p 2^{(\frac{n}{p}-(n-1))j}\big\|f\big\|_{L^p(\mathbf{R}^n)}$$

for all $1 < p \le 2$. When $\frac{n}{n-1} < p \le 2$ the series $\sum_{j=1}^{\infty} 2^{(\frac{n}{p}-(n-1))j}$ converges $(n \ge 3)$ yielding that $\mathscr{M} : L^p \to L^p$ for these p's. The case $\mathscr{M} : L^\infty \to L^\infty$ is easy, while boundedness of $\mathscr{M}$ on L^p for $2 < p < \infty$ follows by interpolating between L^2 and L^∞.

Exercises

6.5.1. Let m be in $L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ that satisfies $|m^\vee(x)| \le C(1+|x|)^{-n-\delta}$ for some $\delta > 0$. Show that the maximal multiplier

$$\mathscr{M}_m(f)(x) = \sup_{t>0}\big|\big(\widehat{f}(\xi)\,m(t\xi)\big)^\vee(x)\big|$$

is L^p bounded for all $1 < p < \infty$.

6.5.2. Suppose that the function m is supported in the annulus $R \le |\xi| \le 2R$ and is bounded by A. Show that the g-function

$$G(f)(x) = \left(\int_0^\infty |(m(t\xi)\widehat{f}(\xi))^\vee(x)|^2\,\frac{dt}{t}\right)^{\frac{1}{2}}$$

maps $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ with bound at most $A\sqrt{\log 2}$.

6.5.3. ([302]) Let $A, a, b > 0$ with $a + b > 1$. Use the idea of Lemma 6.5.2 to show that if $m(\xi)$ satisfies $|m(\xi)| \le A(1+|\xi|)^{-a}$ and $|\nabla m(\xi)| \le A(1+|\xi|)^{-b}$ for all $\xi \in \mathbf{R}^n$, then the maximal operator

$$\mathscr{M}_m(f)(x) = \sup_{t>0}\big|\big(\widehat{f}(\xi)\,m(t\xi)\big)^\vee(x)\big|$$

is bounded from $L^2(\mathbf{R}^n)$ to itself.
[*Hint:* Use that

$$\mathscr{M}_m \le \sum_{j=0}^{\infty} \mathscr{M}_{m,j}\,,$$

where $\mathscr{M}_{m,j}$ corresponds to the multiplier $\varphi_j m$; here φ_j is as in (6.5.8). Show that

$$\big\|\mathscr{M}_{m,j}(f)\big\|_{L^2} \le C\big\|\varphi_j m\big\|_{L^\infty}^{\frac{1}{2}}\big\|\varphi_j\widetilde{m}\big\|_{L^\infty}^{\frac{1}{2}}\big\|f\big\|_{L^2} \le C\,2^{j\frac{1-(a+b)}{2}}\big\|f\big\|_{L^2},$$

where $\widetilde{m}(\xi) = \xi\cdot\nabla m(\xi)$.]