

6.4.3 The Dyadic Martingale Square Function

As a consequence of identity (6.4.7), proved in the previous subsection, we obtain that

$$\left\| \left(\sum_{k \in \mathbf{Z}} |D_k(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)}, \quad (6.4.11)$$

which says that the *dyadic martingale square function*

$$S(f) = \left(\sum_{k \in \mathbf{Z}} |D_k(f)|^2 \right)^{\frac{1}{2}}$$

is L^2 bounded. It is natural to ask whether there exist L^p analogues of this result, and this is the purpose of the following theorem.

Theorem 6.4.7. *For any $1 < p < \infty$ there exists a constant $c_{p,n}$ such that for every function f in $L^p(\mathbf{R}^n)$ we have*

$$\frac{1}{c_{p',n}} \|f\|_{L^p(\mathbf{R}^n)} \leq \|S(f)\|_{L^p(\mathbf{R}^n)} \leq c_{p,n} \|f\|_{L^p(\mathbf{R}^n)}. \quad (6.4.12)$$

The lower inequality *quantifies the growth of the L^p norm of a function f , provided $\|S(f)\|_{L^p(\mathbf{R}^n)} < \infty$.*

Proof. Let $\{r_j\}_j$ be the Rademacher functions (see Appendix C.1) enumerated in such a way that their index set is the set of integers. We rewrite the upper estimate in (6.4.12) as

$$\int_0^1 \int_{\mathbf{R}^n} \left| \sum_{k \in \mathbf{Z}} r_k(\omega) D_k(f)(x) \right|^p dx d\omega \leq C_p^p \|f\|_{L^p}^p. \quad (6.4.13)$$

We prove a stronger estimate than (6.4.13), namely that for all $\omega \in [0, 1]$ we have

$$\int_{\mathbf{R}^n} \left| T_\omega(f)(x) \right|^p dx \leq C_p^p \|f\|_{L^p}^p, \quad (6.4.14)$$

where

$$T_\omega(f)(x) = \sum_{k \in \mathbf{Z}} r_k(\omega) D_k(f)(x).$$

In view of the L^2 estimate (6.4.11), we have that the operator T_ω is L^2 bounded with norm 1. We show that T_ω is weak type $(1, 1)$.

To show that T_ω is of weak type $(1, 1)$ we fix a function $f \in L^1$ and $\alpha > 0$. We apply the Calderón–Zygmund decomposition (Theorem 5.3.1) to f at height α to write

$$f = g + b, \quad b = \sum_j (f - \text{Avg } f) \chi_{Q_j},$$