for all $k \in \mathbf{Z}$. We also define the *dyadic martingale difference operator* D_k as follows:

$$D_k(f) = E_k(f) - E_{k-1}(f),$$

also for $k \in \mathbf{Z}$.

Next we introduce the family of Haar functions.

Definition 6.4.4. For a dyadic interval $I = [m2^{-k}, (m+1)2^{-k})$ we define $I_L = [m2^{-k}, (m+\frac{1}{2})2^{-k})$ and $I_R = [(m+\frac{1}{2})2^{-k}, (m+1)2^{-k})$ to be the left and right parts of I, respectively. The function

$$h_I(x) = |I|^{-\frac{1}{2}}\chi_{I_L} - |I|^{-\frac{1}{2}}\chi_{I_R}$$

is called the *Haar function associated with the interval* I.

We remark that Haar functions are constructed in such a way that they have L^2 norm equal to 1. Moreover, the Haar functions have the following fundamental orthogonality property:

$$\int_{\mathbf{R}} h_I(x) h_{I'}(x)\,dx = \begin{cases} 0 & \text{when } I \neq I', \\ 1 & \text{when } I = I'. \end{cases} \tag{6.4.1}$$

To see this, observe that the Haar functions have L^2 norm equal to 1 by construction. Moreover, if $I \neq I'$, then either $I \cap I' = \emptyset$ or I, I' are related by proper inclusion, say we have $I' \subsetneq I$. Then I' is contained either in the left or in the right half of I, on either of which h_I is constant. Thus (6.4.1) follows.

We recall the notation

$$\langle f, g \rangle = \int_{\mathbf{R}} f(x) g(x)\,dx$$

valid for square integrable functions. Under this notation, (6.4.1) can be rewritten as $\langle h_I, h_{I'} \rangle = \delta_{I,I'}$, where the latter is 1 when $I = I'$ and zero otherwise.

6.4.2 *Relation Between Dyadic Martingale Differences and Haar Functions*

We have the following result relating the Haar functions to the dyadic martingale difference operators in dimension one.

Proposition 6.4.5. *For every locally integrable function f on $\mathbf{R}$ and for all $k \in \mathbf{Z}$ we have the identity*

$$D_k(f) = \sum_{I \in \mathscr{D}_{k-1}} \langle f, h_I \rangle h_I \tag{6.4.2}$$