

[Hint: Majorize this maximal operator by a constant multiple of the sum

$$M(f)(x) + \left( \int_0^\infty |T_t(f)(x) - (f * \varphi_t)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where  $\varphi$  is a  $\mathcal{C}_0^\infty$  function such that  $\widehat{\varphi}(0) = 1$ .]

**6.3.7.** ([150]) Let  $0 < \beta < 1$  and  $p_0 = (1 - \beta/2)^{-1}$ . Suppose that  $\{f_j\}_{j \in \mathbf{Z}}$  are  $L^2$  functions on the real line with norm at most 1. Assume that each  $f_j$  is supported in interval of length 1 and that the orthogonality relation  $|\langle f_j | f_k \rangle| \leq (1 + |j - k|)^{-\beta}$  holds for all  $j, k \in \mathbf{Z}$ .

(a) Let  $I \subseteq \mathbf{Z}$  be such that for all  $j \in I$  the functions  $f_j$  are supported in a fixed interval of length 3. Show that for all  $p$  satisfying  $0 < p \leq 2$  there is  $C_{p,\beta} < \infty$  such that

$$\left\| \sum_{j \in I} \varepsilon_j f_j \right\|_{L^p} \leq C_{p,\beta} |I|^{1 - \frac{\beta}{2}}$$

whenever  $\varepsilon_j$  are complex numbers with  $|\varepsilon_j| \leq 1$ .

(b) Under the same hypothesis as in part (a), prove that for all  $0 < p < p_0$  there is a constant  $C'_{p,\beta} < \infty$  such that

$$\left\| \sum_{j \in I} c_j f_j \right\|_{L^p} \leq C'_{p,\beta} \left( \sum_{j \in \mathbf{Z}} |c_j|^p \right)^{\frac{1}{p}}$$

for all complex-valued sequences  $\{c_j\}_j$  in  $\ell^p$ .

(c) Derive the conclusion of part (b) without the assumption that the  $f_j$  are supported in a fixed interval of length 3.

[Hint: Part (a): Pass from  $L^p$  to  $L^2$  and use the hypothesis. Part (b): Assume  $\sum_{j \in \mathbf{Z}} |c_j|^p = 1$ . For each  $k = 0, 1, \dots$ , set  $I_k = \{j \in \mathbf{Z} : 2^{-k-1} < |c_j| \leq 2^{-k}\}$ . Write  $\left\| \sum_{j \in \mathbf{Z}} c_j f_j \right\|_{L^p} \leq \sum_{k=0}^\infty 2^{-k} \left\| \sum_{j \in I_k} (c_j 2^k) f_j \right\|_{L^p}$ , use part (a), Hölder's inequality, and the fact that  $\sum_{k=0}^\infty 2^{-kp} |I_k| \leq 2^p$ . Part (c): Write  $\sum_{j \in \mathbf{Z}} c_j f_j = \sum_{m \in \mathbf{Z}} F_m$ , where  $F_m$  is the sum of  $c_j f_j$  over all  $j$  such that the support of  $f_j$  meets the interval  $[m, m+1]$ . These  $F_m$ 's are supported in  $[m-1, m+2]$  and are almost orthogonal.]

## 6.4 The Haar System, Conditional Expectation, and Martingales

There is a very strong connection between the Littlewood–Paley operators and certain notions from probability, such as conditional expectation and martingale difference operators. The conditional expectation we are concerned with is with respect to the increasing  $\sigma$ -algebra of all dyadic cubes on  $\mathbf{R}^n$ .