We now show that for all $x \neq 0$, the series $\sum_{j\in\mathbf{Z}} K_j(x)$ converges to a function, which we denote by $K(x)$. Indeed, as a consequence of (6.2.15) we have that

$$(1+2^j\delta)^{\frac{1}{4}} \int_{|x|\geq\delta} |K_j(x)|\,dx \leq \widetilde{C}_n A\,,$$

for any $\delta > 0$, which implies that the function $\sum_{j>0} |K_j(x)|$ is integrable away from the origin and satisfies $\int_{\delta\leq|x|\leq 2\delta} \sum_{j>0} |K_j(x)|\,dx < \infty$. Now note that

$$\int_{\delta\leq|x|\leq 2\delta} |K_j(x)|\,dx \leq \big\|K_j\big\|_{L^2} c_n \delta^{\frac{n}{2}} = \big\|\widehat{\zeta}(2^{-j}\cdot)\,m\big\|_{L^2} c_n \delta^{\frac{n}{2}} \leq c_n \big\|m\big\|_{L^\infty} \delta^{\frac{n}{2}} \big\|\widehat{\zeta}\big\|_{L^2} 2^{\frac{jn}{2}}$$

and from this it follows that $\int_{\delta\leq|x|\leq 2\delta} \sum_{j\leq 0} |K_j(x)|\,dx < \infty$.

We conclude that the series $\sum_{j\in\mathbf{Z}} K_j(x)$ converges a.e. on $\mathbf{R}^n \setminus \{0\}$ to a function $K(x)$ that coincides with the distribution $W = m^\vee$ on $\mathbf{R}^n \setminus \{0\}$ and satisfies

$$\int_{\delta\leq|x|\leq 2\delta} |K(x)|\,dx < \infty.$$

We now prove that the function $K = \sum_{j\in\mathbf{Z}} K_j$ (defined on $\mathbf{R}^n \setminus \{0\}$) satisfies Hörmander's condition. It suffices to prove that for all $y \neq 0$ we have

$$\sum_{j\in\mathbf{Z}} \int_{|x|\geq 2|y|} |K_j(x-y) - K_j(x)|\,dx \leq 2C_n' A\,. \tag{6.2.20}$$

Fix a $y \in \mathbf{R}^n \setminus \{0\}$ and pick a $k \in \mathbf{Z}$ such that $2^{-k} \leq |y| \leq 2^{-k+1}$. The part of the sum in (6.2.20) where $j > k$ is bounded by

$$\begin{aligned}
\sum_{j>k} \int_{|x|\geq 2|y|} |K_j(x-y)| + |K_j(x)|\,dx &\leq 2\sum_{j>k} \int_{|x|\geq|y|} |K_j(x)|\,dx \\
&\leq 2\sum_{j>k} \int_{|x|\geq|y|} |K_j(x)| \frac{(1+2^j|x|)^{\frac{1}{4}}}{(1+2^j|x|)^{\frac{1}{4}}}\,dx \\
&\leq \sum_{j>k} \frac{2\widetilde{C}_n A}{(1+2^j|y|)^{\frac{1}{4}}} \\
&\leq \sum_{j>k} \frac{2\widetilde{C}_n A}{(1+2^j 2^{-k})^{\frac{1}{4}}} = C_n' A\,,
\end{aligned}$$

where we used (6.2.15). The part of the sum in (6.2.20) where $j \leq k$ is bounded by

$$\begin{aligned}
&\sum_{j\leq k} \int_{|x|\geq 2|y|} |K_j(x-y) - K_j(x)|\,dx \\
&\qquad \leq \sum_{j\leq k} \int_{|x|\geq 2|y|} \int_0^1 |-y\cdot\nabla K_j(x-\theta y)|\,d\theta\,dx
\end{aligned}$$

$$\leq \int_0^1 \sum_{j\leq k} 2^{-k+1} \int_{\mathbf{R}^n} |\nabla K_j(x-\theta y)|(1+2^j|x-\theta y|)^{\frac{1}{4}}\,dx\,d\theta$$
$$\leq \int_0^1 \sum_{j\leq k} 2^{-k+1}\widetilde{C}_n A 2^j\,d\theta \leq C_n' A\,,$$

using (6.2.16). Hörmander's condition is satisfied for $K$, and we appeal to Theorem 5.3.3 (in fact the version in the footnote) to complete the proof of (6.2.13). $\square$

**Example 6.2.8.** Let $m$ be a smooth function away from the origin that is homogeneous of imaginary order, i.e., for some fixed $\tau$ real and all $\lambda > 0$ we have

$$m(\lambda\xi) = \lambda^{i\tau} m(\xi)\,. \tag{6.2.21}$$

Then $m$ is an $L^p$ Fourier multiplier for $1 < p < \infty$. Indeed, differentiating both sides of (6.2.21) with respect to $\partial_\xi^\alpha$ we obtain

$$\lambda^{|\alpha|}\partial_\xi^\alpha m(\lambda\xi) = \lambda^{i\tau}\partial_\xi^\alpha m(\xi)$$

and taking $\lambda = |\xi|^{-1}$, we deduce condition (6.2.14) with $C_\alpha = \sup_{|\theta|=1}|\partial^\alpha m(\theta)|$. An explicit example of such a function is $m(\xi) = |\xi|^{i\tau}$. Another example is

$$m_0(\xi_1,\xi_2,\xi_3) = \frac{\xi_1^2+\xi_2^2}{\xi_1^2+i(\xi_2^2+\xi_3^2)}$$

which is homogeneous of degree zero and also smooth on $\mathbf{R}^n\setminus\{0\}$.

**Example 6.2.9.** Let $z$ be a complex numbers with $\mathrm{Re}\,z \geq 0$. Then the functions

$$m_1(\xi) = \left(\frac{|\xi|^2}{1+|\xi|^2}\right)^z, \qquad m_2(\xi) = \left(\frac{1}{1+|\xi|^2}\right)^z$$

defined on $\mathbf{R}^n$ are $L^p$ Fourier multipliers for $1 < p < \infty$. To prove this assertion for $m_1$, we verify condition (6.2.14). To achieve this, introduce the function on $\mathbf{R}^{n+1}$

$$M_1(\xi_1,\dots,\xi_n,t) = \left(\frac{|\xi_1|^2+\cdots+|\xi_n|^2}{t^2+|\xi_1|^2+\cdots+|\xi_n|^2}\right)^z = \left(\frac{|\xi|^2}{t^2+|\xi|^2}\right)^z,$$

where $\xi = (\xi_1,\dots,\xi_n)$. Then $M$ is homogeneous of degree 0 and smooth on $\mathbf{R}^{n+1}\setminus\{0\}$. The derivatives $\partial^\beta M_1$ are homogeneous of degree $-|\beta|$ and by the calculation in the preceding example they satisfy $|\partial^\beta M_1(\xi,t)| \leq C_\beta|(\xi,t)|^{-|\beta|}$, with $C_\beta = \sup_{|\theta|=1}|\partial^\beta M_1(\theta)|$, whenever $(\xi,t)\neq 0$ and $\beta$ is a multi index of $n+1$ variables. In particular, taking $\beta = (\alpha,0)$, we obtain

$$\left|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n} M_1(\xi_1,\dots,\xi_n,t)\right| \leq \frac{C_\alpha}{(t^2+|\xi|^2)^{|\alpha|/2}}\,,$$

and setting $t=1$ we deduce that $|\partial^\alpha m_1(\xi)| \leq C_\alpha(1+|\xi|^2)^{-|\alpha|/2} \leq C_\alpha|\xi|^{-|\alpha|}$.