are $L^p$ multipliers. Since $m'$ is integrable over all intervals of the form $[2^j,\xi]$ when $2^j \le \xi < 2^{j+1}$, the fundamental theorem of calculus gives

$$m(\xi) = m(2^j) + \int_{2^j}^{\xi} m'(t)\,dt, \qquad \text{for } 2^j \le \xi < 2^{j+1},$$

from which it follows that for a Schwartz function $f$ on the real line we have

$$m(\xi)\widehat{f}(\xi)\chi_{I_j^+}(\xi) = m(2^j)\widehat{f}(\xi)\chi_{I_j^+}(\xi) + \int_{2^j}^{2^{j+1}} \widehat{f}(\xi)\chi_{[t,\infty)}(\xi)\chi_{I_j^+}(\xi)\,m'(t)\,dt\,.$$

We therefore obtain the identity

$$(\widehat{f}\chi_{I_j}m_+)^\vee = (\widehat{f}m\chi_{I_j^+})^\vee = m(2^j)\Delta_{I_j^+}(f) + \int_{2^j}^{2^{j+1}} \Delta_{[t,\infty)}\Delta_{I_j^+}(f)\,m'(t)\,dt\,,$$

which implies that

$$|(\widehat{f}\chi_{I_j}m_+)^\vee| \le \|m\|_{L^\infty}|\Delta_{I_j^+}(f)| + A^{\frac{1}{2}}\bigg(\int_{2^j}^{2^{j+1}} \big|\Delta_{[t,\infty)}\Delta_{I_j^+}(f)\big|^2\,|m'(t)|\,dt\bigg)^{\frac{1}{2}},$$

using the hypothesis (6.2.2). Taking $\ell^2(\mathbf{Z})$ norms we obtain

$$\begin{aligned}\Big(\sum_{j\in\mathbf{Z}} |(\widehat{f}\chi_{I_j}m_+)^\vee|^2\Big)^{\frac{1}{2}} \le \|m\|_{L^\infty}\Big(\sum_{j\in\mathbf{Z}} |\Delta_{I_j^+}(f)|^2\Big)^{\frac{1}{2}} \\ + A^{\frac{1}{2}}\bigg(\int_0^\infty \big|\Delta_{[t,\infty)}\Delta^{\#}_{[\log_2 t]}(f_+)\big|^2\,|m'(t)|\,dt\bigg)^{\frac{1}{2}},\end{aligned}$$

where $f_+ = (\widehat{f}\chi_{[0,\infty)})^\vee$. Exercise 5.6.2 gives

$$\begin{aligned}A^{\frac{1}{2}}\bigg\|\bigg(\int_0^\infty \big|\Delta_{[t,\infty)}\Delta^{\#}_{[\log_2 t]}(f_+)\big|^2|m'(t)|\,dt\bigg)^{\frac{1}{2}}\bigg\|_{L^p} \\ \le C\max(p,(p-1)^{-1})A^{\frac{1}{2}}\bigg\|\bigg(\int_0^\infty \big|\Delta^{\#}_{[\log_2 t]}(f_+)\big|^2|m'(t)|\,dt\bigg)^{\frac{1}{2}}\bigg\|_{L^p},\end{aligned}$$

while the hypothesis on $m'$ implies the inequality

$$\bigg\|\Big(\sum_{j\in\mathbf{Z}} \big|\Delta_{I_j^+}(f)\big|^2\int_{I_j^+}|m'(t)|\,dt\Big)^{\frac{1}{2}}\bigg\|_{L^p} \le A^{\frac{1}{2}}\bigg\|\Big(\sum_j |\Delta_{I_j^+}(f)|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p}.$$

Using Theorem 6.1.5 we obtain that

$$\bigg\|\Big(\sum_j |\Delta_{I_j^+}(f)|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p} \le C'\max(p,(p-1)^{-1})^2\big\|(\widehat{f}\chi_{(0,\infty)})^\vee\big\|_{L^p},$$

and the latter is at most a constant multiple of $\max(p,(p-1)^{-1})^3\big\|f\big\|_{L^p}$. Putting things together we deduce that

$$\Big\|\Big(\sum_j|(\widehat{f}\chi_{I_j}m_+)^\vee|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\le C''\max(p,(p-1)^{-1})^4\big(A+\|m\|_{L^\infty}\big)\big\|f\big\|_{L^p}\,, \tag{6.2.4}$$

from which we obtain the estimate

$$\big\|(\widehat{f}m_+)^\vee\big\|_{L^p}\le C\max(p,(p-1)^{-1})^6\big(A+\|m\|_{L^\infty}\big)\big\|f\big\|_{L^p}\,,$$

using the lower estimate of Theorem 6.1.5. This proves (6.2.3) for $m_+$. A similar argument also works for $m_-$, and this concludes the proof by summing the corresponding estimates for $m_+$ and $m_-$. $\square$

We remark that the same proof applies under the more general assumption that $m$ is a function of bounded variation on every interval $[2^j,2^{j+1}]$ and $[-2^{j+1},-2^j]$. In this case the measure $|m'(t)|\,dt$ should be replaced by the ~~total~~ absolute variation $|dm(t)|$ of the Lebesgue–Stieltjes measure $dm(t)$.

**Example 6.2.3.** Any bounded function that is constant on dyadic intervals is an $L^p$ multiplier. Also, the function

$$m(\xi)=|\xi|2^{-[\log_2|\xi|]}$$

is an $L^p$ multiplier on $\mathbf{R}$ for $1<p<\infty$.

## *6.2.2 The Marcinkiewicz Multiplier Theorem on* $\mathbf{R}^n$

We now extend this theorem on $\mathbf{R}^n$. As usual we denote the coordinates of a point $\xi\in\mathbf{R}^n$ by $(\xi_1,\dots,\xi_n)$. We recall the notation $I_j=(-2^{j+1},-2^j]\bigcup[2^j,2^{j+1})$ and $R_{\mathbf{j}}=I_{j_1}\times\cdots\times I_{j_n}$ whenever $\mathbf{j}=(j_1,\dots,j_n)\in\mathbf{Z}^n$.

**Theorem 6.2.4.** *Let $m$ be a bounded function on $\mathbf{R}^n$ such that for all $\alpha=(\alpha_1,\dots,\alpha_n)$ with $|\alpha_1|,\dots,|\alpha_n|\le 1$ the derivatives $\partial^\alpha m$ are continuous up to the boundary of $R_{\mathbf{j}}$ for all $\mathbf{j}\in\mathbf{Z}^n$. Assume that there is a constant $A<\infty$ such that for all partitions $\{s_1,\dots,s_k\}\cup\{r_1,\dots,r_\ell\}=\{1,2,\dots,n\}$ with $n=k+\ell$ and all $\xi\in R_{\mathbf{j}}$ we have*

$$\sup_{\xi_{r_1}\in I_{j_{r_1}}}\cdots\sup_{\xi_{r_\ell}\in I_{j_{r_\ell}}}\int_{I_{j_{s_1}}}\cdots\int_{I_{j_{s_k}}}\big|(\partial_{s_1}\cdots\partial_{s_k}m)(\xi_1,\dots,\xi_n)\big|\,d\xi_{s_k}\cdots d\xi_{s_1}\le A \tag{6.2.5}$$

*for all $\mathbf{j}=(j_1,\dots,j_n)\in\mathbf{Z}^n$. Then $m$ is in $\mathscr{M}_p(\mathbf{R}^n)$ whenever $1<p<\infty$ and there is a constant $C_n<\infty$ such that*

$$\|m\|_{\mathscr{M}_p(\mathbf{R}^n)}\le C_n\big(A+\|m\|_{L^\infty}\big)\max\big(p,(p-1)^{-1}\big)^{6n}. \tag{6.2.6}$$