$F \circ h$ is a holomorphic function on D, $\log|F \circ h|$ is a subharmonic function on D. Applying (1.3.35) to the function $z \mapsto \log|F(h(z))|$, we obtain

$$\log|F(h(z))| \le \frac{1}{2\pi}\int_{-\pi}^{+\pi} \log|F(h(Re^{i\varphi}))| \frac{R^2-\rho^2}{|Re^{i\varphi}-\rho e^{i\theta}|^2}\, d\varphi \tag{1.3.36}$$

when $z = \rho e^{i\theta}$ and $|z| = \rho < R$. For $R < |\zeta| = 1$ the hypothesis on F implies

$$\begin{aligned}
\log|F(h(Re^{i\varphi}))| &\le Ae^{\tau_0\left|\operatorname{Im}\frac{1}{\pi i}\log\left(i\frac{1+R\zeta}{1-R\zeta}\right)\right|} \\
&\le Ae^{\frac{\tau_0}{\pi}\left|\log\left|\frac{1+R\zeta}{1-R\zeta}\right|\right|} \\
&\le A2^{\frac{\tau_0}{\pi}}\left[|1+R\zeta|^{-\frac{\tau_0}{\pi}} + |1-R\zeta|^{-\frac{\tau_0}{\pi}}\right].
\end{aligned}$$

Now $|1 \pm Re^{i\varphi}|^2 = (1 \pm R\cos\varphi)^2 + R^2\sin^2\varphi \ge \frac{1}{4}\sin^2\varphi$, since if $R \le 1/2$ the first term is at least $1/4$ while if $R > 1/2$ the second term in the sum is at least $\frac{1}{4}\sin^2\varphi$. Hence $|1 \pm Re^{i\varphi}| \ge \frac{1}{2}|\sin\varphi|$, thus $\log|F(h(Re^{i\varphi}))| \le C|\sin\varphi|^{-\frac{\tau_0}{\pi}}$. Now $|\sin\varphi|^{-\frac{\tau_0}{\pi}}$ is integrable over $[-\pi,\pi]$, in view of the assumption $\tau_0 < \pi$. Moreover, the bound $\frac{R^2-\rho^2}{|Re^{i\varphi}-\rho e^{i\theta}|^2} \le \frac{4}{1-\rho}$ holds for $1 > R > \frac{1}{2}(\rho+1)$. Fatou's lemma ($\limsup_R \to \infty$) yields

$$\log|F(h(\rho e^{i\theta}))| \le \frac{1}{2\pi}\int_{-\pi}^{+\pi} \log|F(h(e^{i\varphi}))| \frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}\, d\varphi. \tag{1.3.37}$$

Setting $x = h(\rho e^{i\theta})$, we obtain that

$$\rho e^{i\theta} = h^{-1}(x) = \frac{e^{\pi i x}-i}{e^{\pi i x}+i} = -i\frac{\cos(\pi x)}{1+\sin(\pi x)} = \left(\frac{\cos(\pi x)}{1+\sin(\pi x)}\right)e^{-i(\pi/2)},$$

from which it follows that $\rho = (\cos(\pi x))/(1+\sin(\pi x))$ and $\theta = -\pi/2$ when $0 < x \le \frac{1}{2}$, while $\rho = -(\cos(\pi x))/(1+\sin(\pi x))$ and $\theta = \pi/2$ when $\frac{1}{2} \le x < 1$. In either case we easily deduce that

$$\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2} = \frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}.$$

Using this we write (1.3.37) as

$$\log|F(x)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)} \log|F(h(e^{i\varphi}))|\, d\varphi. \tag{1.3.38}$$

On the interval $[-\pi,0)$ we use the change of variables $it = h(e^{i\varphi})$ or, equivalently, $e^{i\varphi} = -\tanh(\pi t) - i\,\mathrm{sech}(\pi t)$. Observe that as φ ranges from $-\pi$ to 0, t ranges from $+\infty$ to $-\infty$. Furthermore, $d\varphi = -\pi\,\mathrm{sech}(\pi t)\, dt$. We have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{0} \frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)} \log|F(h(e^{i\varphi}))|\, d\varphi \\
&\qquad = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\cosh(\pi t)-\cos(\pi x)} \log|F(it)|\, dt.
\end{aligned} \tag{1.3.39}$$

On the interval $(0,\pi]$ we use the change of variables $1+it = h(e^{i\varphi})$ or, equivalently, $e^{i\varphi} = -\tanh(\pi t) + i\,\mathrm{sech}(\pi t)$. Observe that as φ ranges from 0 to π, t ranges from $-\infty$ to $+\infty$. Furthermore, $d\varphi = \pi\,\mathrm{sech}(\pi t)\,dt$. Similarly, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{\pi}&\frac{\sin(\pi t)}{1+\cos(\pi t)\sin(\varphi)}\log|F(h(e^{i\varphi}))|\,d\varphi\\&=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\sin(\pi x)}{\cosh(\pi t)+\cos(\pi x)}\log|F(1+it)|\,dt.\end{aligned}\tag{1.3.40}$$

Adding (1.3.39) and (1.3.40) and using (1.3.38) we conclude the proof when $y=0$.

We now consider the case where $y\neq 0$. Fix $y\neq 0$ and define the function $G(z) = F(z+iy)$. Then G is analytic on the open strip $S=\{z\in\mathbf{C}:\,0<\mathrm{Re}\,z<1\}$ and continuous on its closure. Moreover, for some $A<\infty$ and $0\le\tau_0<\pi$ we have

$$\log|G(z)| = \log|F(z+iy)| \le A\,e^{\tau_0|\mathrm{Im}\,z+y|}\le Ae^{\tau_0|y|}\,e^{\tau_0|\mathrm{Im}\,z|}$$

for all $z\in\overline{S}$. Then the case $y=0$ for G (with A replaced by $Ae^{\tau_0|y|}$) yields

$$|G(x)|\le\exp\left\{\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|G(it)|}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log|G(1+it)|}{\cosh(\pi t)+\cos(\pi x)}\right]dt\right\},$$

which yields the required conclusion for any real y, since $G(x)=F(x+iy)$, $G(it)=F(it+iy)$, and $G(1+it)=F(1+it+iy)$. $\square$

Exercises

1.3.1. Generalize Theorem 1.3.2 to the situation in which T is *quasi-subadditive*, that is, it satisfies for some $K>0$,

$$|T(f+g)|\le K(|T(f)|+|T(g)|),$$

for all f, g in the domain of T. Prove that in this case, the constant A in (1.3.7) can be taken to be K times the constant in (1.3.8).

1.3.2. Let (X,μ), (Y,ν) be two σ-finite measure spaces. Let $1<p<r\le\infty$ and suppose that T be a linear operator defined on the space $L^{p_0}(X)+L^{p_1}(X)$ and taking values in the space of measurable functions on Y. Assume that T maps $L^1(X)$ to $L^{1,\infty}(Y)$ with norm A_0 and $L^r(X)$ to $L^r(Y)$ with norm A_1. Let $0<p_0<p_1\le\infty$. Prove that T maps L^p to L^p with norm at most

$$8\,(p-1)^{-\frac{1}{p}}\,A_0^{\frac{\frac{1}{p}-\frac{1}{r}}{1-\frac{1}{r}}}\,A_1^{\frac{1-\frac{1}{p}}{1-\frac{1}{r}}}.$$

$\big[$*Hint:* First interpolate between L^1 and L^r using Theorem 1.3.2 and then interpolate between $L^{\frac{p+1}{2}}$ and L^r using Theorem 1.3.4.$\big]$