6.2.1 The Marcinkiewicz Multiplier Theorem on R

Proposition 6.2.1 suggests that the behavior of m on each dyadic rectangle R_j should play a crucial role in determining whether m is an L^p multiplier. The Marcinkiewicz multiplier theorem provides such sufficient conditions on m restricted to any dyadic rectangle R_j. Before stating this theorem, we illustrate its main idea via the following example. Suppose that m is a bounded function that vanishes near $-\infty$, that is differentiable at every point, and whose derivative is integrable. Then we may write

$$m(\xi) = \int_{-\infty}^{\xi} m'(t)\,dt = \int_{-\infty}^{+\infty} \chi_{[t,\infty)}(\xi) m'(t)\,dt\,,$$

from which it follows that for a Schwartz function f we have

$$(\widehat{f}m)^{\vee} = \int_{\mathbf{R}} (\widehat{f}\chi_{[t,\infty)})^{\vee} m'(t)\,dt.$$

Since the operators $f \mapsto (\widehat{f}\chi_{[t,\infty)})^{\vee}$ map $L^p(\mathbf{R})$ to itself independently of t, it follows that

$$\big\|(\widehat{f}m)^{\vee}\big\|_{L^p} \le C_p \big\|m'\big\|_{L^1} \big\|f\big\|_{L^p}\,,$$

thus yielding that m is in $\mathscr{M}_p(\mathbf{R})$. The next multiplier theorem is an improvement of this result and is based on the Littlewood–Paley theorem. We begin with the one-dimensional case, which already captures the main ideas.

Theorem 6.2.2. ***(Marcinkiewicz multiplier theorem)*** *Let* $m:\ \mathbf{R} \to \mathbf{R}$ *be a bounded function that is* $\mathscr{C}^1$ *in every dyadic set* $(2^j, 2^{j+1}) \bigcup (-2^{j+1}, -2^j)$ *for* $j \in \mathbf{Z}$. *Assume that the derivative* m' *of* m *satisfies*

$$\sup_j \left[\int_{-2^{j+1}}^{-2^j} |m'(\xi)|\,d\xi + \int_{2^j}^{2^{j+1}} |m'(\xi)|\,d\xi \right] \le A < \infty. \tag{6.2.2}$$

Then for all $1 < p < \infty$ *we have that* $m \in \mathscr{M}_p(\mathbf{R})$ *and for some* $C > 0$ *we have*

$$\big\|m\big\|_{\mathscr{M}_p(\mathbf{R})} \le C \max\big(p, (p-1)^{-1}\big)^6 \big(\big\|m\big\|_{L^\infty} + A\big). \tag{6.2.3}$$

Proof. Since the function m has an integrable derivative on $(2^j, 2^{j+1})$, it has bounded variation in this interval and hence it is a difference of two increasing functions. Therefore, m has left and right limits at the points 2^j and 2^{j+1}, and by redefining m at these points we may assume that m is right continuous at the points 2^j and left continuous at the points -2^j.

Set $I_j = [2^j, 2^{j+1}) \cup (-2^{j+1}, -2^j]$ and $I_j^+ = [2^j, 2^{j+1})$ whenever $j \in \mathbf{Z}$. Given an interval I in $\mathbf{R}$, we introduce an operator Δ_I defined by $\Delta_I(f) = (\widehat{f}\chi_I)^{\vee}$. With this notation, $\Delta_{I_j^+}$ is "half" of the operator $\Delta_j^{\#}$ introduced in (6.1.26). Given m as in the statement of the theorem, we write $m(\xi) = m_+(\xi) + m_-(\xi)$, where $m_+(\xi) = m(\xi)\chi_{\xi \ge 0}$ and $m_-(\xi) = m(\xi)\chi_{\xi < 0}$. We show that both m_+ and m_-