6.1.8. Let m be a bounded function on $\mathbf{R}^n$ that is supported in the annulus $1 \le |\xi| \le 2$ and define $T_j(f) = \big(\widehat{f}(\xi)m(2^{-j}\xi)\big)^\vee$. Suppose that the square function $f \mapsto \big(\sum_{j\in\mathbf{Z}} |T_j(f)|^2\big)^{1/2}$ is bounded on $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$. Show that for every finite subset S of the integers we have

$$\Big\| \sum_{j\in S} T_j(f) \Big\|_{L^p(\mathbf{R}^n)} \le C_{p,n} \big\| f \big\|_{L^p(\mathbf{R}^n)}$$

for some constant $C_{p,n}$ independent of S.

6.1.9. Fix a nonzero Schwartz function h on the line whose Fourier transform is supported in the interval $\big[-\frac{1}{8}, \frac{1}{8}\big]$. For $\{a_j\}$ a sequence of numbers, set

$$f(x) = \sum_{j=1}^{M} a_j e^{2\pi i 2^j x} h(x)\,.$$

Prove that for all $1 < p < \infty$ there exists a constant C_p independent of M such that

$$\|f\|_{L^p(\mathbf{R})} \le C_p \Big(\sum_j |a_j|^2 \Big)^{\frac{1}{2}} \|h\|_{L^p}\,.$$

$\big[$*Hint:* Write $f = \sum_{j=1}^{\infty} \Delta_j(a_j e^{2\pi i 2^j(\cdot)} h)$, where Δ_j is given by convolution with $\varphi_{2^{-j}}$ for some φ whose Fourier transform is supported in the interval $\big[\frac{6}{8}, \frac{10}{8}\big]$ and is equal to 1 on $\big[\frac{7}{8}, \frac{9}{8}\big]$. Then use (6.1.21).$\big]$

6.1.10. Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus $\frac{1}{2} \le |\xi| \le 2$ and that satisfies (6.1.7). Define a Schwartz function Φ by setting

$$\widehat{\Phi}(\xi) = \begin{cases} \sum_{j\le 0} \widehat{\Psi}(2^{-j}\xi) & \text{when } \xi \neq 0, \\ 1 & \text{when } \xi = 0. \end{cases}$$

Let S_0 be the operator given by convolution with Φ. Let $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n)$. Show that

$$\big\| f \big\|_{L^p} \approx \big\| S_0(f) \big\|_{L^p} + \Big\| \Big(\sum_{j=1}^{\infty} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}\,.$$

$\big[$*Hint:* Use Theorem 6.1.2 together with the identity $S_0 + \sum_{j=1}^{\infty} \Delta_j = I$.$\big]$

6.2 Two Multiplier Theorems

We now return to the spaces $\mathscr{M}_p$ introduced in Section 2.5. We seek sufficient conditions on L^∞ functions defined on $\mathbf{R}^n$ to be elements of $\mathscr{M}_p$. In this section we are concerned with two fundamental theorems that provide such sufficient conditions.