Exercises

6.1.1. Construct a Schwartz function Ψ that satisfies $\sum_{j\in\mathbf{Z}}|\widehat{\Psi}(2^{-j}\xi)|^2=1$ for all $\xi\in\mathbf{R}^n\setminus\{0\}$ and whose Fourier transform is supported in the annulus $\frac{6}{7}\le|\xi|\le 2$ and is equal to 1 on the annulus $1\le|\xi|\le\frac{13}{7}$.
$\big[$*Hint:* Set $\widehat{\Psi}(\xi)=\eta(\xi)\big(\sum_{k\in\mathbf{Z}}|\eta(2^{-k}\xi)|^2\big)^{-1/2}$ for a suitable $\eta\in\mathscr{C}_0^\infty(\mathbf{R}^n)$.$\big]$

6.1.2. Suppose that Ψ is an integrable function on $\mathbf{R}^n$ that satisfies $|\widehat{\Psi}(\xi)|\le B\min(|\xi|^{\varepsilon},|\xi|^{-\varepsilon'})$ for some $\varepsilon',\varepsilon>0$. Show that for some constant $C_{\varepsilon,\varepsilon'}<\infty$ we have

$$\sup_{\xi\in\mathbf{R}^n}\bigg(\int_0^\infty|\widehat{\Psi}(t\xi)|^2\frac{dt}{t}\bigg)^{\frac{1}{2}}+\sup_{\xi\in\mathbf{R}^n}\bigg(\sum_{j\in\mathbf{Z}}|\widehat{\Psi}(2^{-j}\xi)|^2\bigg)^{\frac{1}{2}}\le C_{\varepsilon,\varepsilon'}B\,.$$

6.1.3. Let Ψ be an integrable function on $\mathbf{R}^n$ with mean value zero that satisfies

$$|\Psi(x)|\le B(1+|x|)^{-n-\varepsilon},\qquad\int_{\mathbf{R}^n}|\Psi(x-y)-\Psi(x)|\,dx\le B\,|y|^{\varepsilon'},$$

for some $B,\varepsilon',\varepsilon>0$ and for all $y\neq 0$.
(a) Prove that $|\widehat{\Psi}(\xi)|\le c_{n,\varepsilon,\varepsilon'}B\min(|\xi|^{\min(\frac{\varepsilon}{2},1)},|\xi|^{-\varepsilon})$ for some constant $c_{n,\varepsilon,\varepsilon'}$ and conclude that (6.1.4) holds for $p=2$.
(b) Deduce the validity of (6.1.4) and (6.1.5).
(c) If $\varepsilon<1$ and the assumption $|\Psi(x)|\le B(1+|x|)^{-n-\varepsilon}$ is weakened to $|\Psi(x)|\le B|x|^{-n-\varepsilon}$ for all $x\in\mathbf{R}^n$, then show that $|\widehat{\Psi}(\xi)|\le c_{n,\varepsilon,\varepsilon'}B\min(|\xi|^{\frac{\varepsilon}{2}},|\xi|^{-\varepsilon})$ and thus (6.1.4) and (6.1.5) are valid.
$\big[$*Hint:* Part (a): Make use of the identity

$$\widehat{\Psi}(\xi)=\int_{\mathbf{R}^n}e^{-2\pi ix\cdot\xi}\Psi(x)\,dx=-\int_{\mathbf{R}^n}e^{-2\pi ix\cdot\xi}\Psi(x-y)\,dx\,,$$

where $y=\frac{1}{2}\frac{\xi}{|\xi|^2}$ when $|\xi|\ge1$. For $|\xi|\le1$ use the mean value property of Ψ to write $\widehat{\Psi}(\xi)=\int_{\mathbf{R}^n}\Psi(x)(e^{-2\pi ix\cdot\xi}-1)\,dx$ and split the integral in the regions $|x|\le1$ and $|x|\ge1$. Part (b): If $\vec{K}$ is defined by (6.1.13), then control the $\ell^2(\mathbf{Z})$ norm by the $\ell^1(\mathbf{Z})$ norm to prove (6.1.16). Then split the sum $\sum_{j\in\mathbf{Z}}\int_{|x|\ge2|y|}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|\,dx$ into the parts $\sum_{2^j\le|y|^{-1}}$ and $\sum_{2^j>|y|^{-1}}$. Part (c): Notice that when $\varepsilon<1$, we have $|\int_{|x|\le1}\Psi(x)(e^{-2\pi ix\cdot\xi}-1)\,dx|\le C_nB|\xi|^{\frac{\varepsilon}{2}}$. $\big]$

6.1.4. Let Ψ be an integrable function on $\mathbf{R}^n$ with mean value zero that satisfies

$$|\Psi(x)|\le B(1+|x|)^{-n-\varepsilon},\qquad\int_{\mathbf{R}^n}|\Psi(x-y)-\Psi(x)|\,dx\le B|y|^{\varepsilon'},$$

for some $B, \varepsilon' > \varepsilon > 0$ and for all $y \neq 0$. Let $\Psi_t(x) = t^{-n}\Psi(x/t)$. (a) Prove that there are constants c_n, c_n' such that

$$\left(\int_0^\infty |\Psi_t(x)|^2 \frac{dt}{t} \ \cancel{dx}\right)^{\frac{1}{2}} \le c_n B\,|x|^{-n},$$

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \int_{|x|\ge 2|y|} \left(\int_0^\infty |\Psi_t(x-y)-\Psi_t(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}} dx \le c_n' B.$$

(b) Show that there exist constants C_n, C_n' such that for all $1 < p < \infty$ and for all $f \in L^p(\mathbf{R}^n)$ we have

$$\left\|\left(\int_0^\infty |f*\Psi_t|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n)} \le C_n B \max(p,(p-1)^{-1}) \big\|f\big\|_{L^p(\mathbf{R}^n)}$$

and also for all $f \in L^1(\mathbf{R}^n)$ we have

$$\left\|\left(\int_0^\infty |f*\Psi_t|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n' B \big\|f\big\|_{L^1(\mathbf{R}^n)}.$$

(c) Under the additional hypothesis that $0 < \int_0^\infty |\widehat{\Psi}(t\xi)|^2 \frac{dt}{t} = c_0$ for all $\xi \in \mathbf{R}^n \setminus \{0\}$, prove that for all $f \in L^p(\mathbf{R}^n)$ we have

$$\big\|f\big\|_{L^p(\mathbf{R}^n)} \le C_n'' B \max(p,(p-1)^{-1}) \left\|\left(\int_0^\infty |f*\Psi_t|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n)}$$

[*Hint:* Part (a): Use the Cauchy-Schwarz inequality to obtain

$$\int_{|x|\ge 2|y|} \left(\int_0^\infty |\Psi_t(x-y)-\Psi_t(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}} dx$$
$$\le c_n |y|^{-\frac{\varepsilon}{2}} \left(\int_{|x|\ge 2|y|} |x|^{n+\varepsilon} \int_0^\infty |\Psi_t(x-y)-\Psi_t(x)|^2 \frac{dt}{t}\, dx\right)^{\frac{1}{2}},$$

and split the integral on the right into the regions $t \le |y|$ and $t > |y|$. In the second region use that Ψ is bounded to replace the square by the first power. Part (b): Use Exercise 6.1.2 and part (a) of Exercise 6.1.3 and to deduce the inequality when $p = 2$. Then apply Theorem 5.6.1. Part (c): Prove the inequality first for $f \in \mathscr{S}(\mathbf{R}^n)$ using duality.]

6.1.5. Prove the following generalization of Theorem 6.1.2. Let $A > 0$. Suppose that $\{K_j\}_{j\in\mathbf{Z}}$ is a sequence of locally integrable functions on $\mathbf{R}^n \setminus \{0\}$ that satisfies

$$\sup_{x\neq 0} |x|^n \Big(\sum_{j\in\mathbf{Z}} |K_j(x)|^2\Big)^{\frac{1}{2}} \le A,$$

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \int_{|x|\ge 2|y|} \Big(\sum_{j\in\mathbf{Z}} |K_j(x-y)-K_j(x)|^2\Big)^{\frac{1}{2}} dx \le A < \infty,$$