$$\le \Big\| \Big(\sum_{j\in \mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} C_n B \max\big(p', (p'-1)^{-1}\big) \big\|g\big\|_{L^{p'}}, \quad (6.1.20)$$

having used the definition of the adjoint (Section 2.5.2), the Cauchy–Schwarz inequality, Hölder's inequality, and (6.1.4). Taking the supremum over all g in $L^{p'}$ with norm at most one, we obtain that the tempered distribution $f-Q$ is a bounded linear functional on $L^{p'}$. By the Riesz representation theorem, $f-Q$ coincides with an L^p function whose norm satisfies the estimate

$$\big\|f-Q\big\|_{L^p} \le C_n B \max\big(p, (p-1)^{-1}\big) \Big\| \Big(\sum_{j\in \mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}.$$

We now show uniqueness. If Q_1 is another polynomial, with $f-Q_1 \in L^p$, then $Q-Q_1$ must be an L^p function; but the only polynomial that lies in L^p is the zero polynomial. This completes the proof of the converse of the theorem under hypothesis (6.1.6).

To obtain the same conclusion under the hypothesis (6.1.7) we argue in a similar way but we leave the details as an exercise. (One may adapt the argument in the proof of Corollary 6.1.7 to this setting.) □

Remark 6.1.3. We make some observations. If $\widehat{\Psi}$ is real-valued, then the operators Δ_j are self-adjoint. Indeed,

$$\int_{\mathbf{R}^n} \Delta_j(f)\,\overline{g}\,dx = \int_{\mathbf{R}^n} \widehat{f}\,\widehat{\Psi_{2^{-j}}}\,\overline{\widehat{g}}\,d\xi = \int_{\mathbf{R}^n} \widehat{f}\,\overline{\widehat{\Psi_{2^{-j}}}\,\widehat{g}}\,d\xi = \int_{\mathbf{R}^n} f\,\overline{\Delta_j(g)}\,dx.$$

Moreover, if Ψ is a radial function, we see that the operators Δ_j are self-transpose, that is, they satisfy

$$\int_{\mathbf{R}^n} \Delta_j(f)\,g\,dx = \int_{\mathbf{R}^n} f\,\Delta_j(g)\,dx.$$

Assume now that Ψ is either radial or it has a real-valued Fourier transform. Suppose also that Ψ satisfies (6.1.3) and that it has mean value zero. Then the inequality

$$\Big\| \sum_{j\in \mathbf{Z}} \Delta_j(f_j) \Big\|_{L^p} \le C_n B \max\big(p, (p-1)^{-1}\big) \Big\| \Big(\sum_{j\in \mathbf{Z}} |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \quad (6.1.21)$$

is true for sequences of functions $\{f_j\}_j$. To see this we use duality. Let

$$\vec{T}(f) = \{\Delta_j(f)\}_j.$$

Then

$$\vec{T}^*(\{g_j\}_j) = \sum_j \Delta_j(g_j).$$

Inequality (6.1.4) says that the operator $\vec{T}$ maps $L^p(\mathbf{R}^n, \mathbf{C})$ to $L^p(\mathbf{R}^n, \ell^2)$, and its dual statement is that $\vec{T}^*$ maps $L^{p'}(\mathbf{R}^n, \ell^2)$ to $L^{p'}(\mathbf{R}^n, \mathbf{C})$. This is exactly the statement in (6.1.21) if p is replaced by p'. Since p is any number in $(1, \infty)$, (6.1.21) is proved.

6.1.2 Vector-Valued Analogues

We now obtain a vector-valued extension of Theorem 6.1.2. We have the following.

Proposition 6.1.4. *Let Ψ be an integrable $\mathscr{C}^1$ function on $\mathbf{R}^n$ with mean value zero that satisfies (6.1.3) and let Δ_j be the Littlewood–Paley operator associated with Ψ. Then there exists a constant $C_n < \infty$ such that for all $1 < p, r < \infty$ and all sequences of L^p functions f_j we have*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|\Delta_k(f_j)|^2\Big)^{\frac{r}{2}}\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n B\widetilde{C}_{p,r}\Big\|\Big(\sum_{j\in\mathbf{Z}}|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)},$$

where $\widetilde{C}_{p,r} = \max(p,(p-1)^{-1})\max(r,(r-1)^{-1})$. Moreover, for some $C_n' > 0$ and all sequences of L^1 functions f_j we have

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|\Delta_k(f_j)|^2\Big)^{\frac{r}{2}}\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n' B\max(r,(r-1)^{-1})\Big\|\Big(\sum_{j\in\mathbf{Z}}|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1(\mathbf{R}^n)}.$$

In particular,

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n B\widetilde{C}_{p,r}\Big\|\Big(\sum_{j\in\mathbf{Z}}|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)}. \tag{6.1.22}$$

Proof. We introduce Banach spaces $\mathscr{B}_1 = \mathbf{C}$ and $\mathscr{B}_2 = \ell^2$ and for $f \in L^p(\mathbf{R}^n)$ define an operator

$$\vec{T}(f) = \{\Delta_k(f)\}_{k\in\mathbf{Z}}.$$

In the proof of Theorem 6.1.2 we showed that $\vec{T}$ has a kernel $\vec{K}$ that satisfies condition (6.1.16). Furthermore, $\vec{T}$ obviously maps $L^r(\mathbf{R}^n,\mathbf{C})$ to $L^r(\mathbf{R}^n,\ell^2)$. Applying Proposition 5.6.4, we obtain the first two statements of the proposition. Restricting to $k = j$ yields (6.1.22). □

6.1.3 L^p Estimates for Square Functions Associated with Dyadic Sums

Let us pick a Schwartz function Ψ whose Fourier transform is compactly supported in the annulus $2^{-1} \le |\xi| \le 2^2$ such that (6.1.6) is satisfied. (Clearly (6.1.6) has no chance of being satisfied if $\widehat{\Psi}$ is supported only in the annulus $1 \le |\xi| \le 2$.) The Littlewood–Paley operation $f \mapsto \Delta_j(f)$ represents the smoothly truncated frequency localization of a function f near the dyadic annulus $|\xi| \approx 2^j$. Theorem 6.1.2 says that the square function formed by these localizations has L^p norm comparable to that of the original function. In other words, this square function characterizes the L^p norm of a function. This is the main feature of Littlewood–Paley theory.

One may ask whether Theorem 6.1.2 still holds if the Littlewood–Paley operators Δ_j are replaced by their nonsmooth versions

$$f \mapsto \big(\chi_{2^j \le |\xi| < 2^{j+1}} \widehat{f}(\xi)\big)^\vee(x). \tag{6.1.23}$$

This question has a surprising answer that already signals that there may be some fundamental differences between one-dimensional and higher-dimensional Fourier analysis. The square function formed by the operators in (6.1.23) can be used to characterize $L^p(\mathbf{R})$ in the same way Δ_j did, but not $L^p(\mathbf{R}^n)$ when $n > 1$ and $p \ne 2$. The problem lies in the fact that the characteristic function of the unit disk is not an L^p multiplier on $\mathbf{R}^n$ when $n \ge 2$ unless $p = 2$; see Section 5.1 in [131]. The one-dimensional result we alluded to earlier is the following.

For $j \in \mathbf{Z}$ we introduce the one-dimensional operator

$$\Delta_j^\#(f)(x) = (\widehat{f}\chi_{I_j})^\vee(x), \tag{6.1.24}$$

where

$$I_j = [2^j, 2^{j+1}) \cup (-2^{j+1}, -2^j],$$

and $\Delta_j^\#$ is a version of the operator Δ_j in which the characteristic function of the set $2^j \le |\xi| < 2^{j+1}$ replaces the function $\widehat{\Psi}(2^{-j}\xi)$.

Theorem 6.1.5. *There exists a constant C_1 such that for all $1 < p < \infty$ and all f in $L^p(\mathbf{R})$ we have*

$$\frac{\|f\|_{L^p(\mathbf{R})}}{C_1(p+\frac{1}{p-1})^2} \le \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\#(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R})} \le C_1(p+\tfrac{1}{p-1})^2 \|f\|_{L^p(\mathbf{R})}. \tag{6.1.25}$$

Proof. Pick a Schwartz function ψ on the line whose Fourier transform is supported in the set $2^{-1} \le |\xi| \le 2^2$ and is equal to 1 on the set $1 \le |\xi| \le 2$. Let Δ_j be the Littlewood–Paley operators associated with ψ. Observe that $\Delta_j \Delta_j^\# = \Delta_j^\# \Delta_j = \Delta_j^\#$, since $\widehat{\psi}$ is equal to one on the support of $\Delta_j^\#(f)\widehat{\ }$. We now use Exercise 5.6.1(a) to obtain

$$\begin{aligned}
\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\#(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} &= \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\# \Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \\
&\le C \max(p, (p-1)^{-1}) \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \\
&\le CB \max(p, (p-1)^{-1})^2 \|f\|_{L^p},
\end{aligned}$$

where the last inequality follows from Theorem 6.1.2. The reverse inequality for $1 < p < \infty$ follows just like the reverse inequality (6.1.8) of Theorem 6.1.2 by simply replacing the Δ_j's by the $\Delta_j^\#$'s and setting the polynomial Q equal to zero. (There is no need to use the Riesz representation theorem here, just the fact that the L^p norm of f can be realized as the supremum of expressions $|\langle f, g\rangle|$ where g has $L^{p'}$ norm at most 1.) $\square$

There is a higher-dimensional version of Theorem 6.1.5 with dyadic rectangles replacing the dyadic intervals. As has already been pointed out, the higher-dimensional version with dyadic annuli replacing the dyadic intervals is false.

Let us introduce some notation. For $j \in \mathbf{Z}$, we denote by I_j the dyadic set $[2^j, 2^{j+1}) \bigcup (-2^{j+1}, -2^j]$ as in the statement of Theorem 6.1.5. For $j_1, \dots, j_n \in \mathbf{Z}$ define a dyadic rectangle

$$R_{j_1,\dots,j_n} = I_{j_1} \times \cdots \times I_{j_n}$$

in $\mathbf{R}^n$. Actually $R_{j_1,\dots,j_n}$ is not a rectangle but a union of 2^n rectangles; with some abuse of language we still call it a rectangle. For notational convenience we write

$$R_{\mathbf{j}} = R_{j_1,\dots,j_n}, \qquad \text{where } \mathbf{j} = (j_1, \dots, j_n) \in \mathbf{Z}^n.$$

Observe that for different $\mathbf{j}, \mathbf{j}' \in \mathbf{Z}^n$ the rectangles $R_{\mathbf{j}}$ and $R_{\mathbf{j}'}$ have disjoint interiors and that the union of all the $R_{\mathbf{j}}$'s is equal to $\mathbf{R}^n$ minus the coordinate planes $x_j = 0$ for some j. In other words, the family of $R_{\mathbf{j}}$'s, where $\mathbf{j} \in \mathbf{Z}^n$, forms a *tiling* of $\mathbf{R}^n$, which we call the *dyadic decomposition* of $\mathbf{R}^n$. We now introduce operators

$$\Delta_{\mathbf{j}}^{\#}(f)(x) = (\widehat{f}\chi_{R_{\mathbf{j}}})^{\vee}(x), \tag{6.1.26}$$

and we have the following n-dimensional extension of Theorem 6.1.5.

Theorem 6.1.6. *For a Schwartz function ψ on the line with integral zero we define the operator*

$$\Delta_{\mathbf{j}}(f)(x) = \big(\widehat{\psi}(2^{-j_1}\xi_1)\cdots\widehat{\psi}(2^{-j_n}\xi_n)\widehat{f}(\xi)\big)^{\vee}(x), \tag{6.1.27}$$

where $\mathbf{j} = (j_1, \dots, j_n) \in \mathbf{Z}^n$. Then there is a dimensional constant C_n such that

$$\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n} |\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n(p+(p-1)^{-1})^n \|f\|_{L^p(\mathbf{R}^n)}. \tag{6.1.28}$$

Let $\Delta_{\mathbf{j}}^{\#}$ be the operators defined in (6.1.26). Then there exists a positive constant C_n such that for all $1 < p < \infty$ and all $f \in L^p(\mathbf{R}^n)$ we have

$$\frac{\|f\|_{L^p(\mathbf{R}^n)}}{C_n(p+\frac{1}{p-1})^{2n}} \le \Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n} |\Delta_{\mathbf{j}}^{\#}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n(p+\tfrac{1}{p-1})^{2n}\|f\|_{L^p(\mathbf{R}^n)}. \tag{6.1.29}$$

Proof. We first prove (6.1.28). Note that if $\mathbf{j} = (j_1, \dots, j_n) \in \mathbf{Z}^n$, then the operator $\Delta_{\mathbf{j}}$ is equal to

$$\Delta_{\mathbf{j}}(f) = \Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)}(f),$$

where the $\Delta_{j_r}^{(r)}$ are one-dimensional operators given on the Fourier transform by multiplication by $\widehat{\psi}(2^{-j_r}\xi_r)$, with the remaining variables fixed. Inequality in (6.1.28) is a consequence of the one-dimensional case. For instance, we discuss the case $n=2$. Using Proposition 6.1.4, we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}|\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^2)}^p\\
&\qquad=\int_{\mathbf{R}}\Big[\int_{\mathbf{R}}\Big(\sum_{j_1\in\mathbf{Z}}\sum_{j_2\in\mathbf{Z}}|\Delta_{j_1}^{(1)}\Delta_{j_2}^{(2)}(f)(x_1,x_2)|^2\Big)^{\frac{p}{2}}dx_1\Big]dx_2\\
&\qquad\le C^p\max(p,(p-1)^{-1})^p\int_{\mathbf{R}}\Big[\int_{\mathbf{R}}\Big(\sum_{j_2\in\mathbf{Z}}|\Delta_{j_2}^{(2)}(f)(x_1,x_2)|^2\Big)^{\frac{p}{2}}dx_1\Big]dx_2\\
&\qquad= C^p\max(p,(p-1)^{-1})^p\int_{\mathbf{R}}\Big[\int_{\mathbf{R}}\Big(\sum_{j_2\in\mathbf{Z}}|\Delta_{j_2}^{(2)}(f)(x_1,x_2)|^2\Big)^{\frac{p}{2}}dx_2\Big]dx_1\\
&\qquad\le C^{2p}\max(p,(p-1)^{-1})^{2p}\int_{\mathbf{R}}\Big[\int_{\mathbf{R}}|f(x_1,x_2)|^p\,dx_2\Big]dx_1\\
&\qquad= C^{2p}\max(p,(p-1)^{-1})^{2p}\big\|f\big\|_{L^p(\mathbf{R}^2)}^p,
\end{aligned}$$

where we also used Theorem 6.1.2 in the calculation. Higher-dimensional versions of this estimate may easily be obtained by induction.

We now turn to the upper inequality in (6.1.29). We pick a Schwartz function ψ whose Fourier transform is supported in the union $[-4,-1/2]\bigcup[1/2,4]$ and is equal to 1 on $[-2,-1]\bigcup[1,2]$. Then we clearly have

$$\Delta_{\mathbf{j}}^{\#}=\Delta_{\mathbf{j}}^{\#}\Delta_{\mathbf{j}}\,,$$

since $\widehat{\psi}(2^{-j_1}\xi_1)\cdots\widehat{\psi}(2^{-j_n}\xi_n)$ is equal to 1 on the rectangle $R_{\mathbf{j}}$. We now use Exercise 5.6.1(b) and estimate (6.1.28) to obtain

$$\begin{aligned}
\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}^{\#}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}&=\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}^{\#}\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\\
&\le C\max(p,(p-1)^{-1})^n\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\\
&\le CB\max(p,(p-1)^{-1})^{2n}\big\|f\big\|_{L^p}.
\end{aligned}$$

The lower inequality in (6.1.29) for $1<p<\infty$ is proved like inequality (6.1.8) in Theorem 6.1.2. The fundamental ingredient in the proof is that $f=\sum_{\mathbf{j}\in\mathbf{Z}^n}\Delta_{\mathbf{j}}^{\#}\Delta_{\mathbf{j}}^{\#}(f)$ for all Schwartz functions f whose Fourier transform is compactly supported away from the coordinate planes, where the sum is interpreted as the L^2-limit of the sequence of partial sums. Thus the series converges in $\mathscr{S}'$, and pairing with a Schwartz function $\overline{g}$, we obtain the lower inequality in (6.1.29) for Schwartz functions, by applying the steps that prove (6.1.20) (with $Q=0$). To prove the lower inequality in (6.1.29) for a general function $f \in L^p(\mathbf{R}^n)$, we approximate it in L^p by a sequence of Schwartz functions whose Fourier transform is compactly supported away from the coordinate planes. Then both sides of the lower inequality in (6.1.29) for the approximating sequence converge to the corresponding sides of the lower inequality in (6.1.29) for f; the convergence of the sequence of L^p norms of the square functions requires the upper inequality in (6.1.29) that was previously established. This concludes the proof of the theorem. □

Next we observe that if the Schwartz function ψ is suitably chosen, then the reverse inequality in estimate (6.1.28) also holds. More precisely, suppose $\widehat{\psi}(\xi)$ is an even smooth real-valued function supported in the set $\frac{9}{10} \le |\xi| \le \frac{21}{10}$ in $\mathbf{R}$ that satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \qquad \xi \in \mathbf{R}\setminus\{0\}; \tag{6.1.30}$$

then we have the following.

Corollary 6.1.7. *Suppose that ψ satisfies (6.1.30) and let $\Delta_{\mathbf{j}}$ be as in (6.1.27). Let f be an L^p function on $\mathbf{R}^n$ such that the function $\big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}(f)|^2\big)^{\frac{1}{2}}$ is in $L^p(\mathbf{R}^n)$. Then there is a constant C_n that depends only on the dimension and ψ such that the lower estimate*

$$\frac{\big\|f\big\|_{L^p}}{C_n(p+\frac{1}{p-1})^n} \le \bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p} \tag{6.1.31}$$

holds.

Proof. If we had $\sum_{j\in\mathbf{Z}}|\widehat{\psi}(2^{-j}\xi)|^2 = 1$ instead of (6.1.30), then we could apply the method used in the lower estimate of Theorem 6.1.2 to obtain the required conclusion. In this case we provide another argument that is very similar in spirit.

We first prove (6.1.31) for Schwartz functions f. Then the series $\sum_{j\in\mathbf{Z}^n}\Delta_{\mathbf{j}}(f)$ converges in L^2 (and hence in $\mathscr{S}'$) to f. Now let g be another Schwartz function. We express the inner product $\langle f,\overline{g}\rangle$ as the action of the distribution $\sum_{\mathbf{j}\in\mathbf{Z}^n}\Delta_{\mathbf{j}}(f)$ on the test function $\overline{g}$:

$$\begin{aligned}
\big|\langle f,\overline{g}\rangle\big| &= \Big|\Big\langle \sum_{\mathbf{j}\in\mathbf{Z}^n}\Delta_{\mathbf{j}}(f),\overline{g}\Big\rangle\Big| \\
&= \Big|\sum_{\mathbf{j}\in\mathbf{Z}^n}\big\langle \Delta_{\mathbf{j}}(f),\overline{g}\big\rangle\Big| \\
&= \Big|\sum_{\mathbf{j}\in\mathbf{Z}^n}\sum_{\substack{\mathbf{k}=(k_1,\dots,k_n)\in\mathbf{Z}^n\\ \exists r\ |k_r-j_r|\le 1\}}}\big\langle \Delta_{\mathbf{j}}(f),\overline{\Delta_{\mathbf{k}}(g)}\big\rangle\Big| \\
&\le \int_{\mathbf{R}^n}\sum_{\mathbf{j}\in\mathbf{Z}^n}\sum_{\substack{\mathbf{k}=(k_1,\dots,k_n)\in\mathbf{Z}^n\\ \exists r\ |k_r-j_r|\le 1\}}}\big|\Delta_{\mathbf{j}}(f)\big|\,\big|\Delta_{\mathbf{k}}(g)\big|\,dx \\
&\le 3^n\int_{\mathbf{R}^n}\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}\big|\Delta_{\mathbf{j}}(f)\big|^2\Big)^{\frac{1}{2}}\Big(\sum_{\mathbf{k}\in\mathbf{Z}^n}\big|\Delta_{\mathbf{k}}(g)\big|^2\Big)^{\frac{1}{2}}dx
\end{aligned}$$