We clearly have that $\|\vec{K}(x)\|_{\mathbf{C}\to\ell^2} = \big(\sum_j |\Psi_{2^{-j}}(x)|^2\big)^{\frac{1}{2}}$, and to be able to apply Theorem 5.6.1 we need to know that for some constant C_n we have

$$\big\|\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2} \le C_n B\,|x|^{-n}, \tag{6.1.14}$$

$$\lim_{\varepsilon\downarrow 0}\int_{\varepsilon\le|y|\le 1}\vec{K}(y)\,dy = \left\{\int_{|y|\le 1}\Psi_{2^{-j}}(y)\,dy\right\}_{j\in\mathbf{Z}}, \tag{6.1.15}$$

$$\sup_{y\neq 0}\int_{|x|\ge 2|y|}\big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2}\,dx \le C_n B. \tag{6.1.16}$$

Of these, (6.1.14) is easily obtained using (6.1.3), (6.1.15) is straightforward[1], and so we focus on (6.1.16). Since Ψ is a $\mathscr{C}^1$ function, for $|x|\ge 2|y|$ we have

$$\begin{aligned}
&|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)| \\
&\quad\le 2^{(n+1)j}|\nabla\Psi(2^j(x-\theta y))|\,|y| \qquad\qquad \text{for some } \theta\in[0,1],\\
&\quad\le B\,2^{(n+1)j}\big(1+2^j|x-\theta y|\big)^{-(n+1)}|y| \\
&\quad\le B\,2^{nj}\big(1+2^{j-1}|x|\big)^{-(n+1)}2^j|y| \qquad\qquad \text{since } |x-\theta y|\ge \tfrac{1}{2}|x|.
\end{aligned} \tag{6.1.17}$$

We also have that

$$\begin{aligned}
&|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)| \\
&\quad\le 2^{nj}|\Psi(2^j(x-y))|+2^{jn}|\Psi(2^jx)| \\
&\quad\le B\,2^{nj}\big(1+2^j|x|\big)^{-(n+1)}+B2^{jn}\big(1+2^{j-1}|x|\big)^{-(n+1)} \\
&\quad\le 2B\,2^{nj}\big(1+2^{j-1}|x|\big)^{-(n+1)}.
\end{aligned} \tag{6.1.18}$$

Taking the geometric mean of (6.1.17) and (6.1.18), we obtain for any $\gamma\in[0,1]$

$$|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)| \le 2^{1-\gamma}B\,2^{nj}(2^j|y|)^{\gamma}\big(1+2^{j-1}|x|\big)^{-(n+1)}. \tag{6.1.19}$$

Using this estimate, when $|x|\ge 2|y|$, we obtain

$$\begin{aligned}
\big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2} &= \Big(\sum_{j\in\mathbf{Z}}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|^2\Big)^{1/2} \\
&\le \sum_{j\in\mathbf{Z}}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big| \\
&\le 2B\Big(|y|\sum_{2^j<\frac{2}{|x|}}2^{(n+1)j}+|y|^{\frac{1}{2}}\sum_{2^j\ge\frac{2}{|x|}}2^{(n+\frac{1}{2})j}(2^{j-1}|x|)^{-(n+1)}\Big) \\
&\le C_nB\big(|y||x|^{-n-1}+|y|^{\frac{1}{2}}|x|^{-n-\frac{1}{2}}\big),
\end{aligned}$$

where we used (6.1.19) with $\gamma=1$ in the first sum and (6.1.19) with $\gamma=1/2$ in the second sum. Using this bound, we easily deduce (6.1.16) by integrating over the

[1] for the purposes of proving (6.1.15) it may be necessary to assume that there are only finitely many j on the left of (6.1.4) and (6.1.5); the case of all j will then be a consequence of the Lebesgue monotone convergence theorem