$$\leq \sup_{\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_2^*)}\leq 1} \big\|\vec{T}'(G)\big\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_1^*)} \|F\|_{L^p(\mathbf{R}^n,\mathscr{B}_1)}$$
$$\leq C_n\, p\,(A+B_\star)\|F\|_{L^p(\mathbf{R}^n,\mathscr{B}_1)}\,,$$

where we used (5.6.12). This combined with (5.6.10) implies the required conclusion whenever $r<\infty$ and $p\in\big(1,\min(r,2)\big)\cup(r,\infty)$. The remaining $p$'s follow by interpolation (Exercise 5.5.1 or Exercise 5.5.3 (a)). $\square$

### 5.6.2 Applications

We proceed with some applications. An important consequence of Theorem 5.6.1 is the following:

**Corollary 5.6.2.** *Let $A,B>0$ and let $W_j$ be a sequence of tempered distributions on $\mathbf{R}^n$ whose Fourier transforms are uniformly bounded functions (i.e., $|\widehat{W_j}|\leq B$). Suppose that for each $j$, $W_j$ is related as in (5.3.7) to a function $K_j$ on $\mathbf{R}^n\setminus\{0\}$ that satisfies*

$$|K_j(x)|\leq A\,|x|^{-n}\,, \qquad x\neq 0, \tag{5.6.14}$$

$$\lim_{\varepsilon_k\to 0}\int_{1\geq|x|\geq\varepsilon_k} K_j(x)\,dx = L_j\,, \tag{5.6.15}$$

*for some complex constant $L_j$, and*

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\geq 2|y|}\sup_j|K_j(x-y)-K_j(x)|\,dx\leq A\,. \tag{5.6.16}$$

*Then there are constants $C_n,C_n'>0$ such that for all $1<p,r<\infty$ we have*

$$\Big\|\Big(\sum_j|W_j*f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}}\leq C_n'\max(r,(r-1)^{-1})(A+B)\Big\|\Big(\sum_j|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1},$$
$$\Big\|\Big(\sum_j|W_j*f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p}\leq C_n\,c(p,r)(A+B)\Big\|\Big(\sum_j|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p},$$

*where $c(p,r)=\max(p,(p-1)^{-1})\max(r,(r-1)^{-1})$.*

*Proof.* Let $T_j$ be the operator given by convolution with the distribution $W_j$. Clearly $T_j$ is $L^2$ bounded with norm at most $B$. It follows from Theorem 5.3.3 that the $T_j$'s are of weak type $(1,1)$ and also bounded on $L^r$ with bounds at most a dimensional constant multiple of $\max(r,(r-1)^{-1})(A+B)$, uniformly in $j$. We fix $N\in\mathbf{Z}^+$, we set $\mathscr{B}_1=\mathscr{B}_2=\ell^r_N=\ell^r(\{-N,\dots,N\})$, and we define

$$\vec{T}(\{f_j\}_j)=\{W_j*f_j\}_j$$

for $\{f_j\}_j \in L^r(\mathbf{R}^n, \ell^r_N)$. It is immediate to verify that $\vec{T}$ maps $L^r(\mathbf{R}^n, \ell^r_N)$ to itself with norm at most a dimensional constant multiple of $\max(r,(r-1)^{-1})(A+B)$. The kernel of $\vec{T}$ is $\vec{K}$ in $L(\ell^r_N, \ell^r_N)$ defined by

$$\vec{K}(x)(\{t_j\}_j) = \{K_j(x)t_j\}_j, \qquad \{t_j\}_j \in \ell^r_N.$$

Obviously, we have

$$\big\|\vec{K}(x-y) - \vec{K}(x)\big\|_{\ell^r_N \to \ell^r_N} \le \sup_{|j|\le N} |K_j(x-y) - K_j(x)|\,,$$

and therefore condition (5.6.3) holds for $\vec{K}$ as a consequence of (5.6.16). Moreover, (5.6.1) and (5.6.2) with $\vec{K}_0 = \{L_j\}_j$ are also valid for this $\vec{K}$, in view of assumptions (5.6.14) and (5.6.15). Finally, the conclusions with the indices restricted by $|j| \le N$ follow from Theorem 5.6.1, and letting $N \uparrow \infty$ we deduce the claimed estimates. $\square$

If all the $W_j$'s are equal, we obtain the following corollary, which contains in particular the inequality (5.5.16) mentioned earlier.

**Corollary 5.6.3.** *Let $W$ be an element of $\mathscr{S}'(\mathbf{R}^n)$ whose Fourier transform is a function bounded in absolute value by some $B > 0$. Suppose that $W$ is related as in (5.3.7) to a locally integrable function $K$ on $\mathbf{R}^n \setminus \{0\}$ that satisfies*

$$|K(x)| \le A\,|x|^{-n}\,, \qquad x \neq 0,$$

$$\lim_{\varepsilon_k \to 0} \int_{\varepsilon_k \le |x| \le 1} K(x)\,dx = L\,,$$

*and*

$$\sup_{y \in \mathbf{R}^n \setminus \{0\}} \int_{|x|\ge 2|y|} |K(x-y) - K(x)|\,dx \le A\,. \tag{5.6.17}$$

*Let $T$ be the operator given by convolution with $W$. Then there exist constants $C_n, C_n' > 0$ such that for all $1 < p, r < \infty$ we have that*

$$\bigg\|\Big(\sum_j |T(f_j)|^r\Big)^{\frac{1}{r}}\bigg\|_{L^{1,\infty}} \le C_n' \max(r,(r-1)^{-1})(A+B)\bigg\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\bigg\|_{L^1},$$

$$\bigg\|\Big(\sum_j |T(f_j)|^r\Big)^{\frac{1}{r}}\bigg\|_{L^p} \le C_n c(p,r)\,(A+B)\bigg\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\bigg\|_{L^p},$$

*where $c(p,r) = \max(p,(p-1)^{-1})\max(r,(r-1)^{-1})$. In particular, these inequalities are valid for the Hilbert transform and the Riesz transforms.*

Interestingly enough, we can use the very statement of Theorem 5.6.1 to obtain its corresponding vector-valued version.

**Proposition 5.6.4.** *Let let $1 < p, r < \infty$ and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two Banach spaces. Suppose that $\vec{T}$ given by (5.6.4) is a bounded linear operator from $L^r(\mathbf{R}^n, \mathscr{B}_1)$ to $L^r(\mathbf{R}^n, \mathscr{B}_2)$ with norm $B = B(r)$. Also assume that for all $x \in \mathbf{R}^n \setminus \{0\}$, $\vec{K}(x)$ is a* *bounded linear operator from $\mathscr{B}_1$ to $\mathscr{B}_2$ that satisfies conditions (5.6.1) , (5.6.2), (5.6.3) for some $A>0$ and $\vec{K}_0 \in L(\mathscr{B}_1,\mathscr{B}_2)$. Then there exist positive constants $C_n, C_n'$ such that for all $\mathscr{B}_1$-valued functions $F_j$ we have*

$$\Big\|\Big(\sum_j \|\vec{T}(F_j)\|_{\mathscr{B}_2}^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n'(A+B)\Big\|\Big(\sum_j \|F_j\|_{\mathscr{B}_1}^r\Big)^{\frac{1}{r}}\Big\|_{L^1(\mathbf{R}^n)},$$

$$\Big\|\Big(\sum_j \|\vec{T}(F_j)\|_{\mathscr{B}_2}^r\Big)^{\frac{1}{r}}\Big\|_{L^{p}(\mathbf{R}^n)} \le C_n(A+B)c(p)\Big\|\Big(\sum_j \|F_j\|_{\mathscr{B}_1}^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)},$$

*where $c(p)=\max(p,(p-1)^{-1})$.*

*Proof.* Let us denote by $\ell^r(\mathscr{B}_1)$ the Banach space of all $\mathscr{B}_1$-valued sequences $\{u_j\}_j$ that satisfy

$$\big\|\{u_j\}_j\big\|_{\ell^r(\mathscr{B}_1)} = \big(\sum_j \|u_j\|_{\mathscr{B}_1}^r\big)^{\frac{1}{r}} < \infty.$$

Now consider the operator $\vec{S}$ defined on $L^r(\mathbf{R}^n, \ell^r(\mathscr{B}_1))$ by

$$\vec{S}(\{F_j\}_j) = \{\vec{T}(F_j)\}_j\,.$$

It is obvious that $\vec{S}$ maps $L^r(\mathbf{R}^n,\ell^r(\mathscr{B}_1))$ to $L^r(\mathbf{R}^n,\ell^r(\mathscr{B}_2))$ with norm at most $B$. Moreover, $\vec{S}$ has kernel $\widetilde{K}(x)\in L(\ell^r(\mathscr{B}_1),\ell^r(\mathscr{B}_2))$ given by

$$\widetilde{K}(x)(\{u_j\}_j) = \{\vec{K}(x)(u_j)\}_j,$$

where $\vec{K}$ is the kernel of $\vec{T}$. It is not hard to verify that for $x\in\mathbf{R}^n\setminus\{0\}$ we have

$$\big\|\widetilde{K}(x)\big\|_{\ell^r(\mathscr{B}_1)\to\ell^r(\mathscr{B}_2)} = \big\|\vec{K}(x)\big\|_{\mathscr{B}_1\to\mathscr{B}_2},$$

hence for $x\neq y\in\mathbf{R}^n$ we also have

$$\big\|\widetilde{K}(x-y)-\widetilde{K}(x)\big\|_{\ell^r(\mathscr{B}_1)\to\ell^r(\mathscr{B}_2)} = \big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}.$$

Moreover, if we define $\widetilde{K_0}\in L\big(\ell^r(\mathscr{B}_1),\ell^r(\mathscr{B}_2)\big)$ by

$$\widetilde{K_0}(\{u_j\}_j) = \big\{\vec{K}_0(u_j)\big\}_j\,.$$

for $\{u_j\}_j\in\ell^r(\mathscr{B}_1)$, then we have

$$\lim_{k\to\infty}\int_{\varepsilon_k\le|y|\le 1}\widetilde{K}(y)\,dy = \widetilde{K_0}$$

in $L\big(\ell^r(\mathscr{B}_1),\ell^r(\mathscr{B}_2)\big)$.

We conclude that $\widetilde{K}$ satisfies conditions (5.6.1) , (5.6.2), (5.6.3). Hence the operator $\vec{S}$ associate with $\widetilde{K}$ satisfies the conclusion of Theorem 5.6.1, that is, the desired inequalities for $\vec{T}$. □

### 5.6.3 Vector-Valued Estimates for Maximal Functions

Next, we discuss applications of vector-valued inequalities to some nonlinear operators. We fix an integrable function $\Phi$ on $\mathbf{R}^n$ and for $t>0$ define $\Phi_t(x)=t^{-n}\Phi(t^{-1}x)$. We suppose that $\Phi$ satisfies the following *regularity* condition:

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\geq 2|y|}\sup_{t>0}|\Phi_t(x-y)-\Phi_t(x)|\,dx=A_\Phi<\infty. \tag{5.6.18}$$

We consider the maximal operator

$$M_\Phi(f)(x)=\sup_{t>0}|(f*\Phi_t)(x)|$$

defined for $f$ in $L^1+L^\infty$. We are interested in obtaining $L^p$ estimates for $M_\Phi$. We observe that the trivial estimate

$$\big\|M_\Phi(f)\big\|_{L^\infty}\leq\|\Phi\|_{L^1}\|f\|_{L^\infty} \tag{5.6.19}$$

holds when $p=\infty$. It is natural to set

$$\mathscr{B}_1=\mathbf{C}\qquad\text{and}\qquad\mathscr{B}_2=L^\infty(\mathbf{R}^+)$$

and view $M_\Phi$ as the linear operator $f\mapsto\{f*\Phi_\delta\}_{\delta>0}$ that maps $\mathscr{B}_1$-valued functions to $\mathscr{B}_2$-valued functions.

To do this precisely, we fix and $\delta_0>0$. Then for each $x\in\mathbf{R}^n$ we define a bounded linear operator $\vec{K}_\Phi(x)$ from $\mathscr{B}_1=\mathbf{C}$ to $\mathscr{B}_2=L^\infty((\delta_0,\infty))$ by setting for $c\in\mathbf{C}$

$$\vec{K}_\Phi(x)(c)=\{c\,\Phi_\delta(x)\}_{\delta>\delta_0}.$$

Clearly we have

$$\big\|\vec{K}_\Phi(x)\big\|_{\mathbf{C}\to L^\infty((\delta_0,\infty))}=\sup_{\delta>\delta_0}|\Phi_\delta(x)|.$$

Now (5.6.18) implies condition (5.6.2) for the kernel $\vec{K}_\Phi$. Also, if for some $C,\varepsilon>0$, $|\Phi(x)|\leq C(1+|x|)^{-n-\varepsilon}$ for all $x$, then (5.6.1) holds (for some $A<\infty$) since

$$\sup_{\delta>0}|\Phi_\delta(x)|\leq A\,|x|^{-n}.$$

Also condition (5.6.3) holds since

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|y|\leq 1}\Phi_\delta(y)\,dy=\int_{|y|\leq 1}\Phi_\delta(y)\,dy\qquad\text{uniformly in }\delta>\delta_0.$$

We also define a $\mathscr{B}_2$-valued linear operator acting on complex-valued functions on $\mathbf{R}^n$ by

$$\vec{M}_\Phi(f)=f*\vec{K}_\Phi=\{f*\Phi_\delta\}_{\delta>\delta_0}.$$

Obviously $\vec{M}_\Phi$ maps $L^\infty(\mathbf{R}^n,\mathscr{B}_1)$ to $L^\infty(\mathbf{R}^n,\mathscr{B}_2)$ with norm at most $\|\Phi\|_{L^1}$.

Applying Theorem 5.6.1 with $r = \infty$ we obtain for $1 < p < \infty$,

$$\big\|\vec{M}_\Phi(f)\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_2)} \le C_n \max(p,(p-1)^{-1})\big(A_\Phi + \|\Phi\|_{L^1}\big)\big\|f\big\|_{L^p(\mathbf{R}^n)}, \tag{5.6.20}$$

which can be immediately improved to

$$\big\|\vec{M}_\Phi(f)\big\|_{L^r(\mathbf{R}^n,\mathscr{B}_2)} \le C_n \max(1,(r-1)^{-1})\big(A_\Phi + \|\Phi\|_{L^1}\big)\big\|f\big\|_{L^r(\mathbf{R}^n)} \tag{5.6.21}$$

via interpolation with estimate (5.6.19) for all $1 < r < \infty$. At this point we let $\delta_0 \downarrow 0$ via the Lebesgue monotone convergence theorem and we deduce the same estimate with $\delta_0 = 0$.

Next we use estimate (5.6.21) to obtain vector-valued estimates for the sublinear operator $M_\Phi$.

**Corollary 5.6.5.** *Let $\Phi$ be an integrable function on $\mathbf{R}^n$ that satisfies (5.6.18). Then there exist dimensional constants $C_n$ and $C_n'$ such that for all $1 < p,r < \infty$ the following vector-valued inequalities are valid:*

$$\Big\|\Big(\sum_j |M_\Phi(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}} \le C_n' c(r)\big(A_\Phi + \|\Phi\|_{L^1}\big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1}, \tag{5.6.22}$$

*where $c(r) = 1+(r-1)^{-1}$, and*

$$\Big\|\Big(\sum_j |M_\Phi(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \le C_n c(p,r)\big(A_\Phi + \|\Phi\|_{L^1}\big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p}, \tag{5.6.23}$$

*where $c(p,r) = \big(1+(r-1)^{-1}\big)\big(p+(p-1)^{-1}\big)$.*

*Proof.* We set $\mathscr{B}_1 = \mathbf{C}$ and $\mathscr{B}_2 = L^\infty((\delta_0,\infty))$ as before. We use estimate (5.6.21) as a starting point in Proposition 5.6.4, which immediately yields the required conclusions (5.6.22) and (5.6.23). Finally, we let $\delta_0 \downarrow 0$. $\square$

Similar estimates hold for the Hardy–Littlewood maximal operator.

**Theorem 5.6.6.** *For $1 < p,r < \infty$ the Hardy–Littlewood maximal function $M$ satisfies the vector-valued inequalities*

$$\Big\|\Big(\sum_j |M(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}} \le C_n'\big(1+(r-1)^{-1}\big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1}, \tag{5.6.24}$$

$$\Big\|\Big(\sum_j |M(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \le C_n\, c(p,r)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p}, \tag{5.6.25}$$

*where $c(p,r) = \big(1+(r-1)^{-1}\big)\big(p+(p-1)^{-1}\big)$.*

*Proof.* Let us fix a positive radial symmetrically decreasing Schwartz function $\Phi$ on $\mathbf{R}^n$ that satisfies $\Phi(x) \ge 1$ when $|x| \le 1$. Then the Hardy–Littlewood maximal function $M(f)$ is pointwise controlled by a constant multiple of the function $M_\Phi(|f|)$.

In view of Corollary 5.6.5, it suffices to check that for such a $\Phi$, (5.6.18) holds. First observe that in view of the decreasing character of $\Phi$, we have

$$\sup_j |f| * \Phi_{2^j} \le M_\Phi(|f|) \le 2^n \sup_j |f| * \Phi_{2^j}\,,$$

and for this reason we choose to work with the easier dyadic maximal operator

$$M_\Phi^d(f) = \sup_j |f * \Phi_{2^j}|\,.$$

We observe the validity of the simple inequalties

$$2^{-n} M(f) \le \mathcal{M}(f) \le \frac{1}{v_n} M_\Phi(|f|) \le \frac{2^n}{v_n} M_\Phi^d(|f|)\,. \tag{5.6.26}$$

If we can show that

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \int_{|x|\ge 2|y|} \sup_{j\in\mathbf{Z}} |\Phi_{2^j}(x-y) - \Phi_{2^j}(x)|\,dx = C_n < \infty\,, \tag{5.6.27}$$

then (5.6.22) and (5.6.23) are satisfied with $M_\Phi^d$ replacing $M_\Phi$. We therefore turn our attention to (5.6.27). We have

$$\begin{aligned}
&\int_{|x|\ge 2|y|} \sup_{j\in\mathbf{Z}} |\Phi_{2^j}(x-y) - \Phi_{2^j}(x)|\,dx \\
&\quad\le \sum_{j\in\mathbf{Z}} \int_{|x|\ge 2|y|} |\Phi_{2^j}(x-y) - \Phi_{2^j}(x)|\,dx \\
&\quad\le \sum_{2^j>|y|} \int_{|x|\ge 2|y|} \frac{|y|\,|\nabla\Phi\big(\frac{x-\theta y}{2^j}\big)|}{2^{(n+1)j}}\,dx + \sum_{2^j\le|y|} \int_{|x|\ge 2|y|} (|\Phi_{2^j}(x-y)| + |\Phi_{2^j}(x)|)\,dx \\
&\quad\le \sum_{2^j>|y|} \int_{|x|\ge 2|y|} \frac{|y|}{2^{(n+1)j}} \frac{C_N\,dx}{(1+|2^{-j}(x-\theta y)|)^N} + 2\sum_{2^j\le|y|} \int_{|x|\ge|y|} |\Phi_{2^j}(x)|\,dx \\
&\quad\le \sum_{2^j>|y|} \int_{|x|\ge 2|y|} \frac{|y|}{2^{(n+1)j}} \frac{C_N}{(1+|2^{-j-1}x|)^N}\,dx + 2\sum_{2^j\le|y|} \int_{|x|\ge 2^{-j}|y|} |\Phi(x)|\,dx \\
&\quad\le \sum_{2^j>|y|} \int_{|x|\ge 2^{-j}|y|} \frac{|y|}{2^j} \frac{C_N}{(1+|x|)^N}\,dx + 2\sum_{2^j\le|y|} C_N(2^{-j}|y|)^{-N} \\
&\quad\le C_N \sum_{2^j>|y|} \frac{|y|}{2^j} + C_N \\
&\quad\le 3C_N\,,
\end{aligned}$$

where $C_N>0$ depends on $N>n$, $\theta\in[0,1]$, and $|x-\theta y|\ge|x|/2$ when $|x|\ge 2|y|$.

Now apply (5.6.22) and (5.6.23) to $M_\Phi^d$ and use (5.6.26) to obtain the desired vector-valued inequalities. $\square$

**Remark 5.6.7.** Observe that (5.6.24) and (5.6.25) also hold for $r=\infty$. These endpoint estimates can be proved directly by observing that

$$\sup_j M(f_j) \le M(\sup_j |f_j|)\,.$$

The same is true for estimates (5.6.22) and (5.6.23). Finally, estimates (5.6.25) and (5.6.23) also hold for $p=r=\infty$.

## Exercises

**5.6.1.** (a) For all $j\in\mathbf{Z}$, let $I_j$ be an interval in $\mathbf{R}$ and let $T_j$ be the operator given on the Fourier transform by multiplication by the characteristic function of $I_j$. Prove that there exists a constant $C>0$ such that for all $1<p,r<\infty$ and for all square integrable functions $f_j$ on $\mathbf{R}$ we have

$$\Big\|\Big(\sum_j |T_j(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R})} \le C\max\Big(r,\frac{1}{r-1}\Big)\max\Big(p,\frac{1}{p-1}\Big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R})},$$

$$\Big\|\Big(\sum_j |T_j(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}(\mathbf{R})} \le C\max\Big(r,\frac{1}{r-1}\Big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1(\mathbf{R})}.$$

(b) Let $R_j$ be arbitrary rectangles on $\mathbf{R}^n$ with sides parallel to the axes and let $S_j$ be the operators given on the Fourier transform by multiplication by the characteristic functions of $R_j$. Prove that there exists a dimensional constant $C_n<\infty$ such that for all indices $1<p,r<\infty$ and for all square integrable functions $f_j$ in $L^p(\mathbf{R}^n)$ we have

$$\Big\|\Big(\sum_j |S_j(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n\max\Big(r,\frac{1}{r-1}\Big)^n\max\Big(p,\frac{1}{p-1}\Big)^n\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)}.$$

[*Hint:* Part (a): Use Theorem 5.5.1 and the identity $T_j=\frac{i}{2}\big(M^aHM^{-a}-M^bHM^{-b}\big)$, if $I_j$ is $\chi_{(a,b)}$, where $M^a(f)(x)=f(x)e^{2\pi iax}$ and $H$ is the Hilbert transform. Part (b): Apply the result in part (a) in each variable.]

**5.6.2.** Let $(T,d\mu)$ be a $\sigma$-finite measure space. For every $t\in T$, let $R(t)$ be a rectangle in $\mathbf{R}^n$ with sides parallel to the axes such that the map $t\mapsto R(t)$ is measurable. Then there is a constant $C_n>0$ such that for all $1<p<\infty$ and for all families of square integrable functions $\{f_t\}_{t\in T}$ on $\mathbf{R}^n$ such that $t\mapsto f_t(x)$ is measurable for all $x\in\mathbf{R}^n$ we have

$$\Big\|\Big(\int_T |(\widehat{f_t}\chi_{R(t)})^\vee|^2\,d\mu(t)\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C_n\max(p,(p-1)^{-1})^n\Big\|\Big(\int_T |f_t|^2\,d\mu(t)\Big)^{\frac{1}{2}}\Big\|_{L^p},$$

[*Hint:* When $n=1$ reduce matters to an $L^p(L^2(T,d\mu),L^2(T,d\mu))$ inequality for the Hilbert transform, via the hint in the preceding exercise. Verify the inequality $p=2$ and then use Theorem 5.6.1 for the other $p$'s. Obtain the $n$-dimensional inequality by iterating the one-dimensional.]

**5.6.3.** Let $\Phi$ be a function on $\mathbf{R}^n$ that satisfies $\sup_{x\in\mathbf{R}^n}|x|^n|\Phi(x)|\le A$ and

$$\int_{\mathbf{R}^n}|\Phi(x-y)-\Phi(x)|\,dx\le\eta(|y|),\qquad\int_{|x|\ge R}|\Phi(x)|\,dx\le\eta(R^{-1}),$$

for all $R\ge 1$, where $\eta$ is a continuous increasing function on $[0,2]$ that satisfies $\eta(0)=0$ and $\int_0^2\frac{\eta(t)}{t}\,dt<\infty$.
(a) Prove that (5.6.27) holds.
(b) Show that if $\Phi$ lies in $L^1(\mathbf{R}^n)$, then the maximal function $f\mapsto\sup_{j\in\mathbf{Z}}|f*\Phi_{2^j}|$ maps $L^p(\mathbf{R}^n)$ to itself for $1<p\le\infty$.
[*Hint:* Part (a): Modify the calculation in the proof of Theorem 5.6.6. Part (b): Use Theorem 5.6.1 with $r=\infty$.]

**5.6.4.** (a) On $\mathbf{R}$, take $f_j=\chi_{[2^{j-1},2^j]}$ to prove that inequality (5.6.25) fails when $p=\infty$ and $1<r<\infty$.
(b) Again on $\mathbf{R}$, take $N>2$ and $f_j=\chi_{[\frac{j-1}{N},\frac{j}{N}]}$ for $j=1,2,\dots,N$ to prove that (5.6.25) fails when $1<p<\infty$ and $r=1$.

**5.6.5.** Let $K$ be an integrable function on the real line and assume that the operator $f\mapsto f*K$ is bounded on $L^p(\mathbf{R})$ for some $1<p<\infty$. Prove that the vector-valued inequality

$$\Big\|\Big(\sum_j|K*f_j|^q\Big)^{\frac1q}\Big\|_{L^p}\le C_{p,q}\Big\|\Big(\sum_j|f_j|^q\Big)^{\frac1q}\Big\|_{L^p}$$

may fail in general when $q<1$.
[*Hint:* Take $K=\chi_{[-1,1]}$ and $f_j=\chi_{[\frac{j-1}{N},\frac{j}{N}]}$ for $1\le j\le N$.]

**5.6.6.** Let $\{Q_j\}_j$ be a countable collection of cubes in $\mathbf{R}^n$ with disjoint interiors. Let $c_j$ be the center of the cube $Q_j$ and $d_j$ its diameter. For $\varepsilon>0$, define the *Marcinkiewicz function* associated with the family $\{Q_j\}_j$ as follows:

$$M_\varepsilon(x)=\sum_j\frac{d_j^{n+\varepsilon}}{|x-c_j|^{n+\varepsilon}+d_j^{n+\varepsilon}}\,.$$

Prove that for some constants $C_{n,\varepsilon,p}$ and $C_{n,\varepsilon}$ one has

$$\big\|M_\varepsilon\big\|_{L^p}\le C_{n,\varepsilon,p}\Big(\sum_j|Q_j|\Big)^{\frac1p},\qquad p>\frac{n}{n+\varepsilon},$$

$$\big\|M_\varepsilon\big\|_{L^{\frac{n}{n+\varepsilon},\infty}}\le C_{n,\varepsilon}\Big(\sum_j|Q_j|\Big)^{\frac{n+\varepsilon}{n}},$$

and consequently $\int_{\mathbf{R}^n}M_\varepsilon(x)\,dx\le C_{n,\varepsilon}\sum_j|Q_j|$.