is $\mathscr{B}_2$-integrable and the expression

$$\int_{|y|\le 1}(f_i(x-y)-f_i(x))\,\|\vec{K}(y)(u_i)\|_{\mathscr{B}_2}\,dy$$

is a well-defined element of $\mathscr{B}_2$. Also the integral in (5.6.5) is over the compact set $1\le|y|\le|x|+M$, where the ball $B(0,M)$ contains the supports of all f_i, and thus it also converges in $\mathscr{B}_2$, using (5.6.1).

The following vector-valued extension of Theorem 5.3.3 is the main result of this section.

Theorem 5.6.1. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be Banach spaces. Suppose that $\vec{K}(x)$ satisfies* (5.6.1)*,* (5.6.2)*, and* (5.6.3) *for some $A>0$ and $\vec{K}_0\in L(\mathscr{B}_1,\mathscr{B}_2)$. Let $\vec{T}$ be the operator associated with $\vec{K}$ as in (5.6.4). Assume that $\vec{T}$ is a bounded linear operator from $L^r(\mathbf{R}^n,\mathscr{B}_1)$ to $L^r(\mathbf{R}^n,\mathscr{B}_2)$ with norm $B_\star$ for some $1<r\le\infty$. Then $\vec{T}$ has well-defined extensions on $L^p(\mathbf{R}^n,\mathscr{B}_1)$ for all $1\le p<\infty$. Moreover, there exist dimensional constants C_n and C_n' such that*

$$\big\|\vec{T}(F)\big\|_{L^{1,\infty}(\mathbf{R}^n,\mathscr{B}_2)}\le C_n'(A+B_\star)\big\|F\big\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}\tag{5.6.6}$$

for all F in $L^1(\mathbf{R}^n,\mathscr{B}_1)$ and

$$\big\|\vec{T}(F)\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_2)}\le C_n\max\big(p,(p-1)^{-1}\big)(A+B_\star)\big\|F\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_1)}\tag{5.6.7}$$

whenever $1<p<\infty$ and F is in $L^p(\mathbf{R}^n,\mathscr{B}_1)$.

Proof. Although $\vec{T}$ is defined on the entire $L^1(\mathbf{R}^n,\mathscr{B}_1)\cap L^r(\mathbf{R}^n,\mathscr{B}_1)$, it will be convenient to work with its restriction to a smaller dense subspace of $L^1(\mathbf{R}^n,\mathscr{B}_1)$. We make the observation that the space $\mathscr{Q}\otimes\mathscr{B}_1$ of all functions of the form $\sum_{i=1}^m \chi_{R_i}u_i$, where R_i are disjoint dyadic cubes and $u_i\in\mathscr{B}_1$, is dense in $L^1(\mathbf{R}^n,\mathscr{B}_1)$. Indeed, by Proposition 5.5.6 (c) it suffices to approximate a $\mathscr{C}_0^\infty\otimes\mathscr{B}_1$-valued function with a $\mathscr{Q}\otimes\mathscr{B}_1$-valued function. But this is immediate since any function in $\mathscr{C}_0^\infty(\mathbf{R}^n)$ can be approximated in $L^1(\mathbf{R}^n)$ by finite linear combinations of characteristic functions of disjoint dyadic cubes.

Case 1: $r=\infty$. We fix $F=\sum_{i=1}^m\chi_{R_i}u_i$ in $\mathscr{Q}\otimes\mathscr{B}_1$ and we notice that for each $x\in\mathbf{R}^n$ we have $\|F(x)\|_{\mathscr{B}_1}=\sum_{i=1}^m\chi_{R_i}(x)\|u_i\|_{\mathscr{B}_1}$, which is also a finite linear combination of characteristic functions of dyadic cubes. Apply the Calderón-Zygmund decomposition to $\|F\|_{\mathscr{B}_1}$ at height $\gamma\alpha$, where $\gamma=2^{-n-1}B_\star^{-1}$ as in the proof of Theorem 5.3.3. We extract a finite collection of closed dyadic cubes $\{Q_j\}_j$ satisfying $\sum_j|Q_j|\le(\gamma\alpha)^{-1}\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}$ and we define the good function of the decomposition

$$G(x)=\begin{cases}F(x) & \text{for } x\notin\cup_j Q_j\\ |Q_j|^{-1}\int_{Q_j}F(x)\,dx & \text{for } x\in Q_j.\end{cases}$$

Also define the bad function $B(x) = F(x) - G(x)$. Then $B(x) = \sum_j B_j(x)$, where each B_j is supported in Q_j and has mean value zero over Q_j. Moreover,

$$\|G\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \le \|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \tag{5.6.8}$$
$$\|G\|_{L^\infty(\mathbf{R}^n,\mathscr{B}_1)} \le 2^n\gamma\alpha \tag{5.6.9}$$

and $\|B_j\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \le 2^{n+1}\gamma\alpha|Q_j|$, by an argument similar to that given in the proof of Theorem 5.3.1. We only verify (5.6.9). On the cube Q_j, G is equal to the constant $|Q_j|^{-1}\int_{Q_j} F(x)\,dx$, and this is bounded by $2^n\gamma\alpha$. For each $x \in \mathbf{R}^n \setminus \bigcup_j Q_j$ and for each $k = 0,1,2,\dots$ there exists a unique nonselected dyadic cube $Q_x^{(k)}$ of generation k that contains x. Then for each $k \ge 0$, we have

$$\left\| \frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} F(y)\,dy \right\|_{\mathscr{B}_1} \le \frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} \|F(y)\|_{\mathscr{B}_1}\,dy \le \gamma\alpha.$$

The intersection of the closures of the cubes $Q_x^{(k)}$ is the singleton $\{x\}$. Using Corollary 2.1.16, we deduce that for almost all $x \in \mathbf{R}^n \setminus \bigcup_j Q_j$ we have

$$F(x) = \sum_{i=1}^m \chi_{R_i}(x)u_i = \sum_{i=1}^m \lim_{k\to\infty}\left(\frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} \chi_{R_i}(y)\,dy\right)u_i = \lim_{k\to\infty}\frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} F(y)\,dy\,.$$

Since these averages are at most $\gamma\alpha$, we conclude that $\|F\|_{\mathscr{B}_1} \le \gamma\alpha$ almost everywhere on $\mathbf{R}^n \setminus \bigcup_j Q_j$, hence $\|G\|_{\mathscr{B}_1} \le \gamma\alpha$ a.e. on this set. This proves (5.6.9).

By assumption we have

$$\|\vec{T}(G)\|_{L^\infty(\mathbf{R}^n,\mathscr{B}_2)} \le B_\star \|G\|_{L^\infty(\mathbf{R}^n,\mathscr{B}_1)} \le 2^n\gamma\alpha B_\star = \alpha/2\,.$$

Then the set $\{x \in \mathbf{R}^n :\ \|\vec{T}(G)(x)\|_{\mathscr{B}_2} > \alpha/2\}$ is null and we have

$$\left|\{x \in \mathbf{R}^n :\ \|\vec{T}(F)(x)\|_{\mathscr{B}_2} > \alpha\}\right| \le \left|\{x \in \mathbf{R}^n :\ \|\vec{T}(B)(x)\|_{\mathscr{B}_2} > \alpha/2\}\right|.$$

Let $Q_j^* = 2\sqrt{n}\,Q_j$. We have

$$\begin{aligned}
&\left|\{x \in \mathbf{R}^n :\ \|\vec{T}(B)(x)\|_{\mathscr{B}_2} > \alpha/2\}\right| \\
&\quad\le \Big|\bigcup_j Q_j^*\Big| + \Big|\{x \notin \bigcup_j Q_j^* :\ \|\vec{T}(B)(x)\|_{\mathscr{B}_2} > \alpha/2\}\Big| \\
&\quad\le \frac{(2\sqrt{n})^n}{\gamma}\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha} + \frac{2}{\alpha}\int_{(\cup_j Q_j^*)^c} \|\vec{T}(B)(x)\|_{\mathscr{B}_2}\,dx \\
&\quad\le \frac{(2\sqrt{n})^n}{\gamma}\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha} + \frac{2}{\alpha}\sum_j\int_{(Q_j^*)^c} \|\vec{T}(B_j)(x)\|_{\mathscr{B}_2}\,dx,
\end{aligned}$$

since $B = \sum_j B_j$. It suffices to estimate the last sum. Denoting by y_j is the center of the cube Q_j and using the fact that B_j has mean value zero over Q_j, we write

$$
\begin{aligned}
&\sum_j \int_{(Q_j^*)^c} \big\|\vec{T}(B_j)(x)\big\|_{\mathscr{B}_2}\,dx \\
&\quad = \sum_j \int_{(Q_j^*)^c} \bigg\| \int_{Q_j} \big(\vec{K}(x-y)-\vec{K}(x-y_j)\big)(B_j(y))\,dy \bigg\|_{\mathscr{B}_2} dx \\
&\quad \le \sum_j \int_{Q_j} \big\|B_j(y)\big\|_{\mathscr{B}_1} \int_{(Q_j^*)^c} \big\|\vec{K}(x-y)-\vec{K}(x-y_j)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\,dx\,dy \\
&\quad \le \sum_j \int_{Q_j} \big\|B_j(y)\big\|_{\mathscr{B}_1} \int_{|x-y_j|\ge 2|y-y_j|} \big\|\vec{K}(x-y)-\vec{K}(x-y_j)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\,dx\,dy \\
&\quad \le A\sum_j \|B_j\|_{L^1(Q_j,\mathscr{B}_1)} \\
&\quad \le 2^{n+1}A\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)},
\end{aligned}
$$

where we used the fact that $|x-y_j|\ge 2|y-y_j|$ for all $x\notin Q_j^*$ and $y\in Q_j$ and (5.6.2). Consequently,

$$
\begin{aligned}
\big|\big\{x\in\mathbf{R}^n:\ \|\vec{T}(F)(x)\|_{\mathscr{B}_2}>\alpha\big\}\big| &\le \frac{(2\sqrt{n})^n}{\gamma}\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha}+\frac{2}{\alpha}2^{n+1}A\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \\
&= \big((2\sqrt{n})^n 2^{n+2}B_\star + 2^{n+1}A\big)\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha} \\
&\le C_n'\,(A+B_\star)\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha},
\end{aligned}
$$

where $C_n'=(2\sqrt{n})^n 2^{n+1}+2^{n+2}$. Thus $\vec{T}$ has an extension that maps $L^1(\mathbf{R}^n,\mathscr{B}_1)$ to $L^{1,\infty}(\mathbf{R}^n,\mathscr{B}_2)$ with constant $C_n'(A+B_\star)$. By interpolation (Exercise 5.5.3 (b)) it has an extension that satisfies (5.6.7).

Case 2: $1<r<\infty$. We fix $F=\sum_{i=1}^m \chi_{R_i}u_i$ in $\mathscr{Q}\otimes\mathscr{B}_1$ and we notice that for each $x\in\mathbf{R}^n$ we have $\|F(x)\|_{\mathscr{B}_1}=\sum_{i=1}^m \chi_{R_i}(x)\|u_i\|_{\mathscr{B}_1}$. Thus the function $x\mapsto\|F(x)\|_{\mathscr{B}_1}$ is a finite linear combination of characteristic functions of disjoint dyadic cubes. We prove the weak type estimate (5.6.6) by applying the Calderón–Zygmund decomposition to the function $x\mapsto\|F(x)\|_{\mathscr{B}_1}$ defined on $\mathbf{R}^n$. Then we decompose $F=G+B$, where G and B satisfy properties analogous to the case $r=\infty$. The new ingredient in this case is that the set $\big\{x\in\mathbf{R}^n:\ \|\vec{T}(G)(x)\|_{\mathscr{B}_2}>\alpha/2\big\}$ is not null but its measure can be estimated as follows:

$$
\big|\big\{x\in\mathbf{R}^n:\ \|\vec{T}(G)(x)\|_{\mathscr{B}_2}>\alpha/2\big\}\big| \le \left(\frac{2B_\star}{\alpha}\right)^r \|G\|^r_{L^r(\mathbf{R}^n,\mathscr{B}_1)} \le \frac{2B_\star}{\alpha}\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)},
$$

where the first inequality is a consequence of the boundedness of $\vec{T}$ on $L^r(\mathbf{R}^n,\mathscr{B}_1)$ and the second is obtained by combining (5.6.8) and (5.6.9). Combining this estimate for the good function with the one for the bad function obtained in the preceding case, it follows that $\vec{T}$ has an extension that satisfies (5.6.6), i.e., it maps $\vec{T}:\ L^1(\mathbf{R}^n,\mathscr{B}_1)$ to $L^{1,\infty}(\mathbf{R}^n,\mathscr{B}_2)$ with constant $C_n'(A+B_\star)$, where $C_n'=2+(2\sqrt{n})^n 2^{n+1}+2^{n+1}$.