Taking the supremum over all finitely simple functions g on Y with $L^{q'}$ norm less than or equal to one, we conclude the proof when $p,q<\infty$. When $q'<p=\infty$ set $f_z=f$ for all z, suitably modifying the argument; when $p<q'=\infty$ set $g_z=g$. $\square$

We now give an application of Theorem 1.3.4.

Example 1.3.6. One may prove Young's inequality (Theorem 1.2.12) using the Riesz–Thorin interpolation theorem (Theorem 1.3.4). Fix a function g in L^r and let $T(f)=f*g$. Since $T:\ L^1\to L^r$ with norm at most $\|g\|_{L^r}$ and $T:\ L^{r'}\to L^\infty$ with norm at most $\|g\|_{L^r}$, Theorem 1.3.4 gives that T maps L^p to L^q with norm at most the quantity $\|g\|_{L^r}^{\theta}\|g\|_{L^r}^{1-\theta}=\|g\|_{L^r}$, where

$$\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{r'}\quad\text{and}\quad\frac{1}{q}=\frac{1-\theta}{r}+\frac{\theta}{\infty}. \tag{1.3.19}$$

Finally, observe that equations (1.3.19) give (1.2.13).

1.3.3 Interpolation of Analytic Families of Operators

Theorem 1.3.4 can be extended to the case in which the interpolated operators are allowed to vary. In particular, if a family of operators depends analytically on a parameter z, then the proof of this theorem can be adapted to work in this setting.

We describe the setup for this theorem. Let (X,μ) and (Y,ν) be σ-finite measure spaces. Suppose that for every z in the closed strip $\overline{S}=\{z\in\mathbf{C}:\ 0\le \operatorname{Re} z\le 1\}$ there is an associated linear operator T_z defined on the space of finitely simple functions on X and taking values in the space of measurable functions on Y such that

$$\int_Y |T_z(\chi_A)\,\chi_B|\,d\nu<\infty \tag{1.3.20}$$

whenever A and B are subsets of finite measure of X and Y, respectively. The family $\{T_z\}_z$ is said to be *analytic* if for all f,g finitely simple functions we have that the function

$$z\mapsto\int_Y T_z(f)\,g\,d\nu \tag{1.3.21}$$

is analytic in the open strip $S=\{z\in\mathbf{C}:\ 0<\operatorname{Re} z<1\}$ and continuous on its closure. The analytic family $\{T_z\}_z$ is called of *admissible growth* if there is a constant τ_0 with $0\le\tau_0<\pi$ such that for finitely simple functions f on X and g on Y there is constant $C(f,g)$ such that

$$\log\left|\int_Y T_z(f)\,g\,d\nu\right|\le C(f,g)\,e^{\tau_0|\operatorname{Im} z|} \tag{1.3.22}$$

for all z satisfying $0\le\operatorname{Re} z\le 1$. Note that if there is $\tau_0\in(0,\pi)$ such that for all measurable subsets A of X and B of Y of finite measure there is a constant $c(A,B)$ such that

$$\log\left|\int_B T_z(\chi_A)\,d\nu\right|\le c(A,B)\,e^{\tau_0|\operatorname{Im} z|}, \tag{1.3.23}$$

then (1.3.22) holds for $f = \sum_{k=1}^{M} a_k \chi_{A_k}$ and $g = \sum_{j=1}^{N} b_j \chi_{B_j}$ and

$$C(f,g) = \log(MN) + \sum_{k=1}^{M} \sum_{j=1}^{N} \left[c(A_k, B_j) + \log\left(|a_k b_j| + 1 \right) \right].$$

The extension of the Riesz–Thorin interpolation theorem is as follows.

Theorem 1.3.7. *Let T_z be an analytic family of linear operators of admissible growth defined on the space of finitely simple functions of a σ-finite measure space (X,μ) and taking values in the set of measurable functions of another σ-finite measure space (Y,ν). Let $1 \le p_0 \ne p_1 \le \infty$, $1 \le q_0 \ne q_1 \le \infty$, and let M_0 and M_1 be positive functions on the real line such that for some τ_1 with $0 \le \tau_1 < \pi$ we have*

$$\sup_{-\infty<y<+\infty} e^{-\tau_1 |y|} \log M_j(y) < \infty \tag{1.3.24}$$

for $j = 0, 1$. Fix $0 < \theta < 1$ and define p, q by the equations

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.3.25}$$

Suppose that for all finitely simple functions f on X we have

$$\big\|T_{iy}(f)\big\|_{L^{q_0}} \le M_0(y) \big\|f\big\|_{L^{p_0}}, \tag{1.3.26}$$

$$\big\|T_{1+iy}(f)\big\|_{L^{q_1}} \le M_1(y) \big\|f\big\|_{L^{p_1}}. \tag{1.3.27}$$

Then for all finitely simple functions f on X we have

$$\big\|T_\theta(f)\big\|_{L^q} \le M(\theta) \big\|f\big\|_{L^p} \tag{1.3.28}$$

where for $0 < x < 1$

$$M(x) = \exp\left\{ \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] dt \right\}.$$

Thus, by density, T_θ has a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,\nu)$ when p and q are as in (1.3.25).

Note that in view of (1.3.24), the integral defining $M(t)$ converges absolutely. The proof of the previous theorem is based on an extension of Lemma 1.3.5.

Lemma 1.3.8. *Let F be analytic on the open strip $S = \{z \in \mathbf{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on its closure such that for some $A < \infty$ and $0 \le \tau_0 < \pi$ we have*

$$\log |F(z)| \le A\, e^{\tau_0 |\operatorname{Im} z|} \tag{1.3.29}$$

for all $z \in \overline{S}$. Then

$$|F(x+iy)| \le \exp\left\{\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log|F(1+it+iy)|}{\cosh(\pi t)+\cos(\pi x)}\right]dt\right\}$$

whenever $0<x<1$, and y is real.

Assuming Lemma 1.3.8, we prove Theorem 1.3.7.

Proof. Fix $0<\theta<1$ and finitely simple functions f on X and g on Y such that $\|f\|_{L^p}=\|g\|_{L^{q'}}=1$. Note that since $0<\theta<1$ we must have $1<p,q<\infty$. Let

$$f=\sum_{k=1}^{m} a_k e^{i\alpha_k}\chi_{A_k} \quad \text{and} \quad g=\sum_{j=1}^{n} b_j e^{i\beta_j}\chi_{B_j}\,,$$

where $a_k>0$, $b_j>0$, α_k, β_j are real, A_k are pairwise disjoint subsets of X with finite measure, and B_j are pairwise disjoint subsets of Y with finite measure for all k,j. Let $P(z)$, $Q(z)$ be as in (1.3.15) and f_z, g_z as in (1.3.16). Define for $z\in\overline{S}$

$$F(z)=\int_Y T_z(f_z)\,g_z\,d\nu\,. \tag{1.3.30}$$

Linearity gives that

$$F(z)=\sum_{k=1}^{m}\sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k}e^{i\beta_j}\int_Y T_z(\chi_{A_k})(x)\,\chi_{B_j}(x)\,d\nu(x)\,,$$

and conditions (1.3.20) together with the fact that $\{T_z\}_z$ is an analytic family imply that $F(z)$ is a well-defined analytic function on the unit strip that extends continuously to its boundary.

Since $\{T_z\}_z$ is a family of admissible growth, (1.3.23) holds for some $c(A_k,B_j)$ and $\tau_0\in(0,\pi)$ and this combined with the facts that

$$|a_k^{P(z)}| \le (1+a_k)^{\frac{p}{p_0}+\frac{p}{p_1}} \qquad \text{and} \qquad |b_j^{Q(z)}| \le (1+b_j)^{\frac{q'}{q'_0}+\frac{q'}{q'_1}}$$

for all z with $0<\operatorname{Re} z<1$, implies (1.3.29) with τ_0 as in (1.3.23) and

$$A=\log(mn)+\sum_{k=1}^{m}\sum_{j=1}^{n}\left[c(A_k,B_j)+\Big(\frac{p}{p_0}+\frac{p}{p_1}\Big)\log(1+a_k)+\Big(\frac{q'}{q'_0}+\frac{q'}{q'_1}\Big)\log(1+b_j)\right].$$

Thus F satisfies the hypotheses of Lemma 1.3.8. Moreover, the calculations in the proof of Theorem 1.3.4 show that (even when $p_0=\infty$, $q_0=1$, $p_1=\infty$, $q_1=1$)

$$\big\|f_{iy}\big\|_{L^{p_0}}=\big\|f\big\|_{L^p}^{\frac{p}{p_0}}=1=\big\|g\big\|_{L^{q'}}^{\frac{q'}{q'_0}}=\big\|g_{iy}\big\|_{L^{q'_0}} \quad \text{when } y\in\mathbf{R}\,, \tag{1.3.31}$$

$$\big\|f_{1+iy}\big\|_{L^{p_1}}=\big\|f\big\|_{L^p}^{\frac{p}{p_1}}=1=\big\|g\big\|_{L^{q'}}^{\frac{q'}{q'_1}}=\big\|g_{1+iy}\big\|_{L^{q'_1}} \quad \text{when } y\in\mathbf{R}\,. \tag{1.3.32}$$