(b) Assume that F lies in a dense subspace of $\mathscr{B}_1$, it satisfies $\|\|F\|_{\mathscr{B}_1}\|_{L^p} = 1$, and it takes only finitely many values. For a $\lambda > 1$ pick a large integer N such that $\lambda^{-N} < \|F(x)\|_{\mathscr{B}_1} \le \lambda^N$ for all $x \in X$ such that $\|F(x)\|_{\mathscr{B}_1} \neq 0$ and define $F_j = F\chi_{\Omega_j}$, where $\Omega_j = \{x : \lambda^j < \|F\|_{\mathscr{B}_1} \le \lambda^{j+1}\}$. Let $a = \frac{p}{p_1} - \frac{p}{p_0}$. Prove the inequalities

$$\Big\|\sum_j \lambda^{-ja\theta} F_j\Big\|^{p_0}_{L^{p_0}(X,\mathscr{B}_1)} \le \lambda^{a\theta p_0} \qquad \text{and} \qquad \Big\|\sum_j \lambda^{ja(1-\theta)} F_j\Big\|^{p_1}_{L^{p_1}(X,\mathscr{B}_1)} \le 1\,.$$

(c) Define $g_0(y) = \max_j \lambda^{-ja\theta}\|\vec{T}(F_j)(y)\|_{\mathscr{B}_2}$, $g_1(y) = \max_j \lambda^{ja(1-\theta)}\|\vec{T}(F_j)(y)\|_{\mathscr{B}_2}$ for $y \in Y$ and show that

$$\|g_0\|_{L^{q_0}(Y)} \le A_0\lambda^{a\theta} \qquad \text{and} \qquad \|g_1\|_{L^{q_1}(Y)} \le A_1\,.$$

(d) Prove that for all $y \in Y$ we have

$$\|\vec{T}(F)(y)\|_{\mathscr{B}_2} \le \sum_j \|\vec{T}(F_j)(y)\|_{\mathscr{B}_2} \le g_0(y)^{1-\theta} g_1(y)^{\theta}\Big(2 + \frac{1}{\lambda^{a\theta}-1} + \frac{1}{\lambda^{a(1-\theta)}-1}\Big)$$

and conclude that $\|\vec{T}(F)\|_{L^p(Y,\mathscr{B}_2)} \le c_\theta A_0^{1-\theta}A_1^{\theta}$ by picking $\lambda = (1+\sqrt{2})^{1/a}$.
$\big[$*Hint:* Part (d): Split the sum according to whether $\lambda^{ja} > \frac{g_1(y)}{g_0(y)}$ and $\lambda^{ja} \le \frac{g_1(y)}{g_0(y)}$.$\big]$

5.5.2. Prove the following version of the Riesz–Thorin interpolation theorem. Let (X,μ) and (Y,ν) be σ-finite measure spaces. Let $1 < p_0, q_0, , p_1, q_1, r_0, s_0, r_1, s_1 < \infty$ and $0 < \theta < 1$ satisfy

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}, \qquad \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{q},$$
$$\frac{1-\theta}{r_0} + \frac{\theta}{r_1} = \frac{1}{r}, \qquad \frac{1-\theta}{s_0} + \frac{\theta}{s_1} = \frac{1}{s}.$$

Suppose that T is a linear operator that maps $L^{p_0}(X)$ to $L^{q_0}(Y)$ and $L^{p_1}(X)$ to $L^{q_1}(Y)$. Define a vector-valued operator $\vec{T}$ by setting $\vec{T}(\{f_j\}_j) = \{T(f_j)\}_j$ acting on sequences of complex-valued functions defined on X. Suppose that $\vec{T}$ maps $L^{p_0}(X,\ell^{r_0}(\mathbf{C}))$ to $L^{q_0}(Y,\ell^{s_0}(\mathbf{C}))$ with norm M_0 and $L^{p_1}(X,\ell^{r_1}(\mathbf{C}))$ to $L^{q_1}(Y,\ell^{s_1}(\mathbf{C}))$ with norm M_1. Prove that $\vec{T}$ maps $L^p(X,\ell^r(\mathbf{C}))$ to $L^q(Y,\ell^s(\mathbf{C}))$ with norm at most $M_0^{1-\theta}M_1^{\theta}$.
$\big[$*Hint:* Use the idea of the proof of Theorem 1.3.4. Apply Lemma 1.3.5 to the function

$$F(z) = \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{l\in\mathbf{Z}} \frac{a_{k,l}^{P(z)} e^{i\alpha_{k,l}}}{\|\{a_{k,l}\}_l\|_{\ell^r}^{R(z)-P(z)}} \frac{b_{j,l}^{Q(z)} e^{i\beta_{j,l}}}{\|\{b_{k,l}\}_l\|_{\ell^{s'}}^{S(z)-Q(z)}} \int_Y T(\chi_{A_k})(y)\,\chi_{B_j}(y)\,d\nu(y)$$