The space $L^p(X)\otimes\ell^r$ is the set of all finite sums

$$\sum_{j=1}^{m}(a_{j1},a_{j2},a_{j3},\dots)\,g_j\,,$$

where $g_j\in L^p(X)$ and $(a_{j1},a_{j2},a_{j3},\dots)\in\ell^r$, $j=1,\dots,m$. This is certainly a subspace of $L^p(X,\ell^r)$. Now given $(f_1,f_2,\dots)\in L^p(X,\ell^r)$, let $F_m=e_1f_1+\cdots+e_mf_m$, where e_j is the infinite sequence with zeros everywhere except at the jth entry, where it has 1. Then $F_m\in L^p(X)\otimes\ell^r$ and approximates f in the norm of $L^p(X,\ell^r)$. This shows the density of $L^p(X)\otimes\ell^r$ in $L^p(X,\ell^r)$.

If T is a linear operator bounded from $L^p(X)$ to $L^q(Y)$, then $\vec{T}$ is defined by

$$\vec{T}(\{f_j\}_j)=\{T(f_j)\}_j.$$

According to Definition 5.5.8, T has a bounded ℓ^r-extension if and only if the inequality

$$\Big\|\Big(\sum_j|T(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^q}\le C\Big\|\Big(\sum_j|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p}$$

is valid.

A linear operator T acting on measurable functions is called *positive* if it satisfies $f\ge 0\implies T(f)\ge 0$. It is straightforward to verify that positive operators satisfy

$$\begin{aligned}&f\le g\implies T(f)\le T(g)\,,\\&|T(f)|\le T(|f|)\,,\\&\sup_j|T(f_j)|\le T\big(\sup_j|f_j|\big)\,,\end{aligned}\tag{5.5.26}$$

for all f,g,f_j measurable functions. We have the following result regarding vector-valued extensions of positive operators:

Proposition 5.5.10. *Let $0<p,q\le\infty$ and (X,μ), (Y,ν) be two σ-finite measure spaces. Let T be a positive linear operator mapping $L^p(X)$ to $L^q(Y)$ (respectively, to $L^{q,\infty}(Y)$) with norm A. Let $\mathscr{B}$ be a Banach space. Then T has a $\mathscr{B}$-valued extension $\vec{T}$ that maps $L^p(X,\mathscr{B})$ to $L^q(Y,\mathscr{B})$ (respectively, to $L^{q,\infty}(Y,\mathscr{B})$) with the same norm.*

Proof. Let us first understand this theorem when $\mathscr{B}=\ell^r$ for $1\le r\le\infty$. The two endpoint cases $r=1$ and $r=\infty$ can be checked easily using the properties in (5.5.26). For instance, for $r=1$ we have

$$\Big\|\sum_j|T(f_j)|\Big\|_{L^q}\le\Big\|\sum_j T(|f_j|)\Big\|_{L^q}=\Big\|T\Big(\sum_j|f_j|\Big)\Big\|_{L^q}\le A\Big\|\sum_j|f_j|\Big\|_{L^p},$$

while for $r=\infty$ we have

$$\Big\|\sup_j|T(f_j)|\Big\|_{L^q}\le\Big\|T(\sup_j|f_j|)\Big\|_{L^q}\le A\,\big\|\sup_j|f_j|\big\|_{L^p}\,.$$