Observe that for every $F \in L^1(X)\otimes\mathscr{B}$ we have

$$\begin{aligned}\Big\|\int_X F(x)\,d\mu(x)\Big\|_{\mathscr{B}} &= \sup_{\|u^*\|_{\mathscr{B}^*}\le 1}\left|\Big\langle u^*,\sum_{j=1}^m\Big(\int_X f_j\,d\mu\Big)u_j\Big\rangle\right|\\ &= \sup_{\|u^*\|_{\mathscr{B}^*}\le 1}\left|\int_X\big\langle u^*,\sum_{j=1}^m f_ju_j\big\rangle\,d\mu\right|\\ &\le \int_X \sup_{\|u^*\|_{\mathscr{B}^*}\le 1}\big|\big\langle u^*,\sum_{j=1}^m f_ju_j\big\rangle\big|\,d\mu\\ &= \big\|F\big\|_{L^1(X,\mathscr{B})}.\end{aligned}$$

Thus the linear operator

$$F\mapsto I_F=\int_X F(x)\,d\mu(x)$$

is bounded from $L^1(X)\otimes\mathscr{B}$ into $\mathscr{B}$. Since every element of $L^1(X,\mathscr{B})$ is a (norm) limit (Proposition 5.5.6 (a)) of a sequence of elements in $L^1(X)\otimes\mathscr{B}$, by continuity, the operator $F\mapsto I_F$ has a unique extension on $L^1(X,\mathscr{B})$ that we call the *Bochner integral* of F and denote by

$$\int_X F(x)\,d\mu(x).$$

$L^1(X,\mathscr{B})$ is called the space of all Bochner integrable functions from X to $\mathscr{B}$. Since the Bochner integral is an extension of I_F, for each $F\in L^1(X,\mathscr{B})$ we have

$$\Big\|\int_X F(x)\,dx\Big\|_{\mathscr{B}}\le\int_X\big\|F(x)\big\|_{\mathscr{B}}\,dx.$$

Consequently, measurable functions F with $\int_X\|F(x)\|_{\mathscr{B}}\,dx<\infty$ are Bochner integrable over X. It is not difficult to show that the Bochner integral of F is the only element of $\mathscr{B}$ that satisfies

$$\Big\langle u^*,\int_X F(x)\,d\mu(x)\Big\rangle=\int_X\big\langle u^*,F(x)\big\rangle\,d\mu(x) \tag{5.5.21}$$

for all $u^*\in\mathscr{B}^*$. The next result concerns duality in this context when $X=\mathbf{R}^n$.

Proposition 5.5.7. *Let $\mathscr{B}$ be a Banach space and $1\le p\le\infty$.*
(a) For any $F\in L^p(\mathbf{R}^n,\mathscr{B})$ we have

$$\|F\|_{L^p(\mathbf{R}^n,\mathscr{B})}=\sup_{\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)}\le 1}\left|\int_{\mathbf{R}^n}\langle G(x),F(x)\rangle\,dx\right|.$$

Consequently, $L^p(\mathbf{R}^n,\mathscr{B})$ isometrically embeds in $(L^{p'}(\mathbf{R}^n,\mathscr{B}^*))^*$.
(b) for any $G\in L^{p'}(\mathbf{R}^n,\mathscr{B}^*)$ one has

$$\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)}=\sup_{\|F\|_{L^p(\mathbf{R}^n,\mathscr{B})}\le 1}\left|\int_{\mathbf{R}^n}\langle G(x),F(x)\rangle\,dx\right|$$

and thus $L^{p'}(\mathbf{R}^n,\mathscr{B}^*)$ isometrically embeds in $(L^p(\mathbf{R}^n,\mathscr{B}))^*$.