to write $I_2(\xi) = J_1(\xi) + J_2(\xi) + J_3(\xi) + J_4(\xi) + J_5(\xi)$, where

$$J_1(\xi) = +\frac{1}{2} \int_{|\xi|^{-1}<|x-z|<N} \big(K(x) - K(x-z)\big) e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.8}$$

$$J_2(\xi) = +\frac{1}{2} \int_{\substack{|\xi|^{-1}<|x|<N \\ |x-z|\le|\xi|^{-1}}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.9}$$

$$J_3(\xi) = +\frac{1}{2} \int_{\substack{|\xi|^{-1}<|x|<N \\ |x-z|\ge N}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.10}$$

$$J_4(\xi) = -\frac{1}{2} \int_{\substack{|\xi|^{-1}<|x-z|<N \\ |x|\le|\xi|^{-1}}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.11}$$

$$J_5(\xi) = -\frac{1}{2} \int_{\substack{|\xi|^{-1}<|x-z|<N \\ |x|\ge N}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,. \tag{5.4.12}$$

Since $2|z| = |\xi|^{-1}$, $J_1(\xi)$ is bounded in absolute value by $\frac{1}{2}A_2$, in view of (5.4.2).

Next observe that $|\xi|^{-1} \le |x| \le \frac{3}{2}|\xi|^{-1}$ in (5.4.9), while $\frac{1}{2}|\xi|^{-1} \le |x| \le |\xi|^{-1}$ in (5.4.11); hence both J_2 and J_4 are bounded by $\frac{1}{2}A_1$. Finally, we have $\frac{1}{2}N < |x| < N$ in (5.4.10) (since $|x| > N - \frac{1}{2}|\xi|^{-1}$), and similarly we have $N \le |x| < \frac{3}{2}N$ in (5.4.12). Thus both J_3 and J_5 are bounded above by $\frac{1}{2}A_1$.

Case 2: If $\varepsilon < N \le |\xi|^{-1}$, then we write

$$\int_{\varepsilon<|x|<N} K(x)e^{-2\pi i x\cdot\xi}\, dx = \int_{\varepsilon<|x|<N} K(x)\, dx + \int_{\varepsilon<|x|<N} K(x)(e^{-2\pi i x\cdot\xi} - 1)\, dx$$

which is bounded in absolute value by

$$A_3 + 2\pi|\xi| \int_{|x|\le|\xi|^{-1}} |K(x)|\, |x|\, dx \le A_3 + 4\pi A_1\,.$$

Notice that if $\xi = 0$, only the first term appears which is bounded by A_3.

Case 3: If $|\xi|^{-1} \le \varepsilon < N$, we write

$$\int_{\varepsilon<|x|<N} K(x)e^{-2\pi i x\cdot\xi} dx = \int_{|\xi|^{-1}<|x|<N} K(x)e^{-2\pi i x\cdot\xi} dx - \int_{|\xi|^{-1}<|x|<\varepsilon} K(x)e^{-2\pi i x\cdot\xi} dx,$$

and each of the terms on the right is similar to $I_2(\xi)$ which was shown to be bounded by $2A_1 + \frac{1}{2}A_2$.

In all cases (5.4.4) holds. □