defines a distribution in $\mathscr{S}'(\mathbf{R}^n)$ if and only if the following limit exists:

$$\lim_{j\to\infty}\int_{\delta_j\le|x|\le 1} K(x)\,dx\,.$$

5.3.3. Suppose that a function K on $\mathbf{R}^n\setminus\{0\}$ satisfies condition (5.3.4) with constant A_1 and condition (5.3.12) with constant A_2.
(a) Show that the functions $K(x)\chi_{|x|\ge\varepsilon}$ also satisfy condition (5.3.12) uniformly in $\varepsilon>0$ with constant A_1+A_2.
(b) Obtain the same conclusion for the upper truncations $K(x)\chi_{|x|\le N}$.
(c) Deduce a similar conclusion for the double truncations $K^{(\varepsilon,N)}(x)=K(x)\chi_{\varepsilon\le|x|\le N}$.

5.3.4. Modify the proof of Theorem 5.3.5 to prove that if $T^{(**)}$ maps L^r to $L^{r,\infty}$ for some $1<r<\infty$, and K satisfies condition (5.3.12), then $T^{(**)}$ maps L^1 to $L^{1,\infty}$.

5.3.5. Assume that T is a linear operator acting on measurable functions on $\mathbf{R}^n$ such that whenever a function f is supported in a cube Q, then $T(f)$ is supported in a fixed multiple of Q.
(a) Suppose that T maps L^p to itself for some $1<p<\infty$ with norm B. Prove that T extends to a bounded operator from L^1 to $L^{1,\infty}$ with norm a constant multiple of B.
(b) Suppose that T maps L^p to L^q for some $1<q<p<\infty$ with norm B. Prove that T extends to a bounded operator from L^1 to $L^{s,\infty}$ with norm a multiple of B, where

$$\frac{1}{p'}+\frac{1}{q}=\frac{1}{s}\,.$$

5.3.6. (a) Let $1<q<\infty$. Show that the good function g in Theorem 5.3.1 lies in the Lorentz space $L^{q,1}$ and that $\|g\|_{L^{q,1}}\le C_{n,q}\alpha^{1/q'}\|f\|_{L^1}^{1/q}$ for some constant $C_{n,q}$.
(b) Let $1<r<\infty$. Obtain a generalization of Theorem 5.3.3 in which the assumption that T maps L^r to L^r is replaced by that T maps $L^{r,1}$ to $L^{r,\infty}$ with norm B.
(c) Let $1<r<\infty$. Obtain a further generalization of Theorem 5.3.3 in which the assumption that T maps L^r to L^r is replaced by that it is of restricted weak type (r,r), i.e., it satisfies

$$|\{x:\ |T(\chi_E)(x)|>\alpha\}|\le B^r\,\frac{|E|}{\alpha^r}$$

for all subsets E of $\mathbf{R}^n$ with finite measure.

5.3.7. Let K satisfy (5.3.12) for some $A_2>0$, let $W\in\mathscr{S}'(\mathbf{R}^n)$ be an extension of K on $\mathbf{R}^n$ as in (5.3.7), and let T be the operator given by convolution with W. Obtain the case $r=\infty$ in Theorem 5.3.3. Precisely, prove that if T maps $L^\infty(\mathbf{R}^n)$ to itself with constant B, then T has an extension on L^1 that satisfies

$$\big\|T\big\|_{L^1\to L^{1,\infty}}\le C_n'\,(A_2+B),$$