The value of the previous theorem lies in the following: Suppose we know that for some sequences $\varepsilon_j \downarrow 0$, $N_j \uparrow \infty$ the pointwise limit $T^{(\varepsilon_j,N_j)}(f)$ exists a.e. for all f in a dense subclass of L^1, then Theorem 5.3.5 and Theorem 2.1.14 allow us to deduce that $T^{(\varepsilon_j,N_j)}(f)$ exists a.e. for all f in $L^1(\mathbf{R}^n)$.

If the singular integrals have kernels of the form $\Omega(x/|x|)|x|^{-n}$ with Ω in L^∞, such as the Hilbert transform and the Riesz transforms, then the upper truncations are not needed for K in (5.3.17). In this case

$$T_\Omega^{(\varepsilon)}(f)(x) = \int_{|y|\ge\varepsilon} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy$$

is well defined for $f \in \bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$ by Hölder's inequality and is equal to

$$\lim_{N\to\infty}\int_{\varepsilon\le|y|\le N} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy\,.$$

Corollary 5.3.6. *The maximal Hilbert transform $H^{(*)}$ and the maximal Riesz transforms $R_j^{(*)}$ are weak type $(1,1)$. Secondly, $\lim_{\varepsilon\to 0} H^{(\varepsilon)}(f)$ and $\lim_{\varepsilon\to 0} R_j^{(\varepsilon)}(g)$ exist a.e. for all $f \in L^1(\mathbf{R})$ and $g \in L^1(\mathbf{R}^n)$, as $\varepsilon \to 0$.*

Proof. Since the kernels $1/x$ on $\mathbf{R}$ and $x_j/|x|^{n+1}$ on $\mathbf{R}^n$ satisfy (5.3.10), the first statement in the corollary is an immediate consequence of Theorem 5.3.5. The second statement follows from Theorem 2.1.14 and Corollary 5.2.8, since these limits exist for Schwartz functions. $\square$

Corollary 5.3.7. *Under the hypotheses of Theorem 5.3.5, $T^{(**)}$ maps $L^p(\mathbf{R}^n)$ to itself for $1<p<2$ with norm*

$$\big\|T^{(**)}\big\|_{L^p\to L^p} \le \frac{C_n(A_1+A_2+B)}{p-1}\,,$$

where C_n is some dimensional constant.

Exercises

5.3.1. Let A_1 be defined in (5.3.4). Prove that

$$\frac{1}{2}A_1 \le \sup_{R>0}\frac{1}{R}\int_{|x|\le R}|K(x)|\,|x|\,dx \le 2A_1\,;$$

thus the expressions in (5.3.6) and (5.3.4) are equivalent.

5.3.2. Suppose that K is a locally integrable function on $\mathbf{R}^n\setminus\{0\}$ that satisfies (5.3.4). Suppose that $\delta_j \downarrow 0$. Prove that the principal value operation

$$\langle W,\varphi\rangle = \lim_{j\to\infty}\int_{\delta_j\le|x|\le 1} K(x)\varphi(x)\,dx$$