operators (often via a linearization process) but it does not apply to quasi-linear operators without some loss in the constant.

Recall that a simple function is called finitely simple if it is supported in a set of finite measure. Finitely simple functions are dense in $L^p(X,\mu)$ for $0<p<\infty$, whenever (X,μ) is a σ-finite measure space.

Theorem 1.3.4. *Let (X,μ) and (Y,ν) be two σ-finite measure spaces. Let T be a linear operator defined on the set of all finitely simple functions on X and taking values in the set of measurable functions on Y. Let $1\le p_0,p_1,q_0,q_1\le\infty$ and assume that*

$$\begin{aligned} \big\|T(f)\big\|_{L^{q_0}} &\le M_0\big\|f\big\|_{L^{p_0}}\,, \\ \big\|T(f)\big\|_{L^{q_1}} &\le M_1\big\|f\big\|_{L^{p_1}}\,, \end{aligned} \tag{1.3.12}$$

for all finitely simple functions f on X. Then for all $0<\theta<1$ we have

$$\big\|T(f)\big\|_{L^q}\le M_0^{1-\theta}M_1^{\theta}\big\|f\big\|_{L^p} \tag{1.3.13}$$

for all finitely simple functions f on X, where

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\qquad \textit{and} \qquad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}\,. \tag{1.3.14}$$

Consequently, when $p<\infty$, by density, T has a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,\nu)$ when p and q are as in (1.3.14).

Proof. If $p=q'=\infty$ there is nothing to prove. Assume first $p,q'<\infty$. Let

$$f=\sum_{k=1}^{m}a_k e^{i\alpha_k}\chi_{A_k}$$

be a finitely simple function on X, where $a_k>0$, α_k are real, and A_k are pairwise disjoint subsets of X with finite measure.

We need to control

$$\big\|T(f)\big\|_{L^q(Y,\nu)}=\sup_g\left|\int_Y T(f)(y)g(y)\,d\nu(y)\right|,$$

where the supremum is taken over all finitely simple functions g on Y with $L^{q'}$ norm less than or equal to 1. Write

$$g=\sum_{j=1}^{n}b_j e^{i\beta_j}\chi_{B_j}\,,$$

where $b_j>0$, β_j are real, and B_j are pairwise disjoint subsets of Y with finite ν-measure. Let

$$P(z)=\frac{p}{p_0}(1-z)+\frac{p}{p_1}z \qquad \text{and} \qquad Q(z)=\frac{q'}{q_0'}(1-z)+\frac{q'}{q_1'}z\,. \tag{1.3.15}$$

For z in the closed strip $\overline{S} = \{z \in \mathbf{C} : 0 \le \operatorname{Re} z \le 1\}$, define

$$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k}, \quad g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j}, \tag{1.3.16}$$

and

$$F(z) = \int_Y T(f_z)(y)\, g_z(y)\, d\nu(y).$$

Notice that $f_\theta = f$ and $g_\theta = g$. By linearity we have

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int_Y T(\chi_{A_k})(y)\, \chi_{B_j}(y)\, d\nu(y).$$

Since $a_k, b_j > 0$, F is analytic in z, and the expression

$$\int_Y T(\chi_{A_k})(y)\, \chi_{B_j}(y)\, d\nu(y)$$

is a finite constant, being an absolutely convergent integral; this is seen by Hölder's inequality with exponents q_0 and q_0' (or q_1 and q_1') and (1.3.12).

By the disjointness of the sets A_k we have (even when $p_0 = \infty$)

$$\big\|f_{it}\big\|_{L^{p_0}} = \big\|f\big\|_{L^p}^{\frac{p}{p_0}},$$

since $|a_k^{P(it)}| = a_k^{\frac{p}{p_0}}$, and by the disjointness of the B_j's we have (even when $q_0 = 1$)

$$\big\|g_{it}\big\|_{L^{q_0'}} = \big\|g\big\|_{L^{q'}}^{\frac{q'}{q_0'}},$$

since $|b_j^{Q(it)}| = b_j^{\frac{q'}{q_0'}}$. Thus Hölder's inequality and the hypothesis give

$$\begin{aligned} |F(it)| &\le \big\|T(f_{it})\big\|_{L^{q_0}} \big\|g_{it}\big\|_{L^{q_0'}} \\ &\le M_0 \big\|f_{it}\big\|_{L^{p_0}} \big\|g_{it}\big\|_{L^{q_0'}} \\ &= M_0 \big\|f\big\|_{L^p}^{\frac{p}{p_0}} \big\|g\big\|_{L^{q'}}^{\frac{q'}{q_0'}}. \end{aligned} \tag{1.3.17}$$

By similar calculations, which are valid even when $p_1 = \infty$ and $q_1 = 1$, we have

$$\big\|f_{1+it}\big\|_{L^{p_1}} = \big\|f\big\|_{L^p}^{\frac{p}{p_1}}$$

and

$$\big\|g_{1+it}\big\|_{L^{q_1'}} = \big\|g\big\|_{L^{q'}}^{\frac{q'}{q_1'}}.$$