Use Corollary 2.1.12 to deduce that

$$\sup_{\varepsilon>0}\left|f*\left((K_{\varepsilon^{-1}})^{(1)}-K_{\varepsilon^{-1}}*\varphi\right)_{\varepsilon}\right|\le c\,(A_1+A_2+A_3)M(f)\,.$$

Finally, take the supremum over $\varepsilon>0$ in (5.3.23) and use (5.3.24) and Corollary 2.1.12 one more time to deduce the estimate

$$T^{(*)}(f)\le M(f*W)+C\,(A_1+A_2+A_3)M(f)\,,$$

where C depends on n and δ; this concludes the proof of (5.3.22) for all functions $f\in\mathscr{S}(\mathbf{R}^n)$. Thus $T^{(*)}$ is bounded on L^p, $1<p<\infty$, when restricted to Schwartz functions.

Now given a general function g in $L^p(\mathbf{R}^n)$ we find a sequence h_j in $\mathscr{S}(\mathbf{R}^n)$ such that $\|h_j-g\|_{L^p}\to 0$ as $j\to\infty$. Then we have the pointwise estimate

$$|T^{(\varepsilon)}(g)|\le|T^{(\varepsilon)}(g-h_j)|+|T^{(\varepsilon)}(h_j)|\le c_{p,n}A_1\varepsilon_0^{-\frac{n}{p}}\|g-h_j\|_{L^p}+|T^{(*)}(h_j)|$$

for all $\varepsilon\ge\varepsilon_0$. Taking the supremum over $\varepsilon\ge\varepsilon_0$ and then L^p norm over the ball $B(0,R)$, we obtain

$$\Big\|\sup_{\varepsilon\ge\varepsilon_0}|T^{(\varepsilon)}(g)|\Big\|_{L^p(B(0,R))}\le c'_{p,n}A_1\varepsilon_0^{-\frac{n}{p}}R^{\frac{n}{p}}\|g-h_j\|_{L^p}+C'\,(A_1+A_2+A_3)\|h_j\|_{L^p}\,.$$

Now we let $j\to\infty$ first, and then $R\to\infty$ and $\varepsilon_0\to 0$ to deduce the boundedness of $T^{(*)}$ on $L^p(\mathbf{R}^n)$ via the Lebesgue monotone convergence theorem.

The assertion concerning the boundedness of $T^{(*)}$ on $L^p(\mathbf{R}^n)$ is a consequence of Theorem 5.4.1, proved in the next section, which claims that if K satisfies conditions (5.3.19), (5.3.20), and (5.3.21), then the Fourier transform of the associated distribution W is in $L^\infty(\mathbf{R}^n)$ and then T is bounded on $L^2(\mathbf{R}^n)$; From this it follows it is also bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$, in view of Theorem 5.3.3. □

5.3.5 *Boundedness for Maximal Singular Integrals Implies Weak Type* $(1,1)$ *Boundedness*

We now state and prove a result analogous to that in Theorem 5.3.3 for maximal singular integrals.

Theorem 5.3.5. *Let $K(x)$ be function on $\mathbf{R}^n\setminus\{0\}$ satisfying (5.3.4) with constant $A_1<\infty$ and Hörmander's condition (5.3.12) with constant $A_2<\infty$. Suppose that the operator $T^{(**)}$ as defined in (5.3.18) maps $L^2(\mathbf{R}^n)$ to itself with norm B. Then $T^{(**)}$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with norm*

$$\big\|T^{(**)}\big\|_{L^1\to L^{1,\infty}}\le C_n(A_1+A_2+B),$$

where C_n is some dimensional constant.

Proof. The proof of this theorem is only a little more involved than the proof of Theorem 5.3.3. We fix an $L^1(\mathbf{R}^n)$ function f. We apply the Calderón–Zygmund decomposition of f at height $\gamma\alpha$ for some $\gamma,\alpha>0$. We then write $f=g+b$, where