for all $f \in \mathscr{S}(\mathbf{R}^n)$, where M is the Hardy–Littlewood maximal operator. Moreover, $T^{(*)}$ is bounded on $L^p(\mathbf{R}^n)$ when $1 < p < \infty$.

Proof. Let φ be a radially decreasing smooth function with integral 1 supported in the ball $B(0,1/2)$. For a function g and $\varepsilon > 0$ we use the notation $g_\varepsilon(x) = \varepsilon^{-n} g(\varepsilon^{-1}x)$. For a distribution W we define W_ε analogously, i.e. as the unique distribution with the property $\langle W_\varepsilon, \psi\rangle = \varepsilon^{-n}\langle W, \psi_{\varepsilon^{-1}}\rangle$. We begin by observing that $K_{\varepsilon^{-1}}(x) = \varepsilon^n K(\varepsilon x)$ satisfies (5.3.19), (5.3.20), and (5.3.21) uniformly in $\varepsilon > 0$.

Set, as before, $K^{(\varepsilon)}(x) = K(x)\chi_{|x|\geq\varepsilon}$. Fix $f \in \mathscr{S}(\mathbf{R}^n)$ for some $1 < p < \infty$. Obviously we have

$$f * K^{(\varepsilon)} = f * \big((K_{\varepsilon^{-1}})^{(1)}\big)_\varepsilon = f * W * \varphi_\varepsilon + f * \big((K_{\varepsilon^{-1}})^{(1)} - W_{\varepsilon^{-1}} * \varphi\big)_\varepsilon . \tag{5.3.23}$$

Next we prove the following estimate for all $\varepsilon > 0$:

$$\Big|\big((K_{\varepsilon^{-1}})^{(1)} - W_{\varepsilon^{-1}} * \varphi\big)(x)\Big| \leq C(A_1 + A_2 + A_3)(1+|x|)^{-n-\delta} \tag{5.3.24}$$

for all $x \in \mathbf{R}^n$. Indeed, for $|x| \geq 1$ we express the left-hand side in (5.3.24) as

$$\left| \int_{\mathbf{R}^n} \big(K_{\varepsilon^{-1}}(x) - K_{\varepsilon^{-1}}(x-y)\big)\varphi(y)\,dy \right| .$$

Since φ is supported in $|y| \leq 1/2$, we have $|x| \geq 2|y|$, and condition (5.3.20) yields that the expression on the left-hand side of (5.3.24) is bounded by

$$\frac{A_2}{|x|^{n+\delta}} \int_{\mathbf{R}^n} |y|^\delta |\varphi(y)|\,dy \leq c\,\frac{A_2}{(1+|x|)^{n+\delta}} ,$$

which proves (5.3.24) in the case $|x| \geq 1$. When $|x| < 1$, the left-hand side of (5.3.24) equals

$$(W_{\varepsilon^{-1}} * \varphi)(x) = \lim_{\delta_j \to 0} \int_{|x-y|\geq\delta_j} K_{\varepsilon^{-1}}(x-y)\varphi(y)\,dy \tag{5.3.25}$$

for some sequence $\delta_j \downarrow 0$; see the discussion in Section 5.3.2. The expression in (5.3.25) is equal to

$$I_1 + I_2 + I_3 ,$$

where

$$\begin{aligned} I_1 &= \int_{|x-y|>\frac18} K_{\varepsilon^{-1}}(x-y)\varphi(y)\,dy , \\ I_2 &= \int_{|x-y|\leq\frac18} K_{\varepsilon^{-1}}(x-y)\big(\varphi(y) - \varphi(x)\big)\,dy , \\ I_3 &= \varphi(x) \lim_{\delta_j \to 0} \int_{\frac18 \geq |x-y| \geq \delta_j} K_{\varepsilon^{-1}}(x-y)\,dy . \end{aligned}$$

In I_1 we have $1/8 \leq |x-y| \leq 1 + 1/2 = 3/2$; hence I_1 is bounded by a multiple of A_1. Since $|\varphi(x) - \varphi(y)| \leq c|x-y|$, the same is valid for I_2. Finally, I_3 is bounded by a multiple of A_3. Combining these facts yields the proof of (5.3.24) in the case $|x| < 1$ as well.