$$T^{(**)}(f) = \sup_{0<\varepsilon<N<\infty} |T^{(\varepsilon,N)}(f)|. \qquad (5.3.18)$$

Since for $f \in L^p(\mathbf{R}^n)$ the functions $T^{(\varepsilon,N)}(f)$ are only defined a.e., it is possible that $T^{(**)}(f)(x)$ is undefined for all $x \in \mathbf{R}^n$. In the sequel, when we write $T^{(**)}(f)(x)$, we will always assume that $T^{(\varepsilon,N)}(f)(x)$ are defined for all $\varepsilon, N > 0$. Moreover, when we write $T^{(**)}(f)$, we will also assume that this function is defined a.e. (for instance this happens when ε, N are restricted to be positive rational numbers.)

One observation is that under condition (5.3.16), one can also define $T^{(*)}(g)$ for general integrable functions g with compact support. In this case, say that the ball $B(0,R)$ contains the support of g. Let $x \in B(0,M)$ and $N = M + R$. Then $|T^{(\varepsilon)}(g)(x)| \leq |g| * |K^{(\varepsilon,N)}|(x)$, which is finite a.e. as the convolution of two L^1 functions; consequently, the integral defining $T^{(\varepsilon)}(g)(x)$ converges absolutely for all $x \in B(0,R)$. Since $R > 0$ is arbitrary, $T^{(\varepsilon)}(g)(x)$ is defined and is finite for ~~almost~~ all $x \in \mathbf{R}^n$.

Obviously $T^{(*)}$ and $T^{(**)}$ are related. If K satisfies condition (5.3.15), then

$$\left| \int_{\varepsilon \leq |y|} f(x-y)K(y)\,dy \right| \leq \sup_{N>0} \left| \int_{\varepsilon \leq |y| \leq N} f(x-y)K(y)\,dy \right|,$$

which implies that

$$T^{(*)}(f) \leq T^{(**)}(f)$$

for all $f \in \bigcup_{1 \leq p < \infty} L^p$. Also, $T^{(\varepsilon,N)}(f) = T^{(\varepsilon)}(f) - T^{(N)}(f)$; hence

$$T^{(**)}(f) \leq 2\,T^{(*)}(f).$$

Therefore, for kernels satisfying (5.3.15), $T^{(**)}$ and $T^{(*)}$ are comparable and the boudnedness properties of $T^{(**)}$ and $T^{(*)}$ are equivalent

Theorem 5.3.4. *(Cotlar's inequality)* *Let $0 < A_1, A_2, A_3 < \infty$ and suppose that K is defined on $\mathbf{R}^n \setminus \{0\}$ and satisfies the size condition,*

$$|K(x)| \leq A_1 |x|^{-n}, \qquad x \neq 0, \qquad (5.3.19)$$

the smoothness condition

$$|K(x-y) - K(x)| \leq A_2 |y|^\delta |x|^{-n-\delta}, \qquad (5.3.20)$$

whenever $|x| \geq 2|y| > 0$, and the cancellation condition

$$\sup_{0<r<R<\infty} \left| \int_{r<|x|<R} K(x)\,dx \right| \leq A_3. \qquad (5.3.21)$$

Let W in $\mathscr{S}'(\mathbf{R}^n)$ be related to K via (5.3.7) and let T be the operator given by convolution with W. Then there is a constant $C_{n,\delta}$ such that the following inequality is valid:

$$T^{(*)}(f) \leq M(T(f)) + C_{n,\delta}(A_1 + A_2 + A_3)M(f), \qquad (5.3.22)$$