$$
\begin{aligned}
&= \frac{2^r}{\alpha^r}\big\|T(g)\big\|_{L^r}^r + \Big|\bigcup_i Q_i^*\Big| + \Big|\Big\{x \notin \bigcup_i Q_i^* : \ |\sum_j T(b_j)(x)| > \frac{\alpha}{2}\Big\}\Big| \\
&\le \frac{2^r}{\alpha^r} B^r \big\|g\big\|_{L^r}^r + \sum_i |Q_i^*| + \frac{2}{\alpha}\int_{(\bigcup_i Q_i^*)^c} \sum_j |T(b_j)(x)|\,dx \\
&\le \frac{2^r}{\alpha^r} 2^{\frac{nr}{r'}} B^r (\gamma\alpha)^{\frac{r}{r'}} \big\|f\big\|_{L^1} + (2\sqrt{n})^n \frac{\big\|f\big\|_{L^1}}{\gamma\alpha} + \frac{2}{\alpha} 2^{n+1} A_2 \big\|f\big\|_{L^1} \\
&\le \left(\frac{(2^{n+1}B\gamma)^r}{2^n\gamma} + \frac{(2\sqrt{n})^n}{\gamma} + 2^{n+2}A_2\right)\frac{\big\|f\big\|_{L^1}}{\alpha}.
\end{aligned}
$$

Choosing $\gamma = 2^{-(n+1)}B^{-1}$, we deduce the weak type $(1,1)$ estimate (5.3.13) for T with $C_n = 2 + 2^{n+1}(2\sqrt{n})^n + 2^{n+2}$.

It follows from Exercise 1.3.2 that T has an extension that maps L^p to L^p with bound at most $C_n'(A_2 + B)(p-1)^{-1/p}$ when $1 < p < r$. The adjoint operator T^* of T, defined by

$$\big\langle T(f)\,|\,g\big\rangle = \big\langle f\,|\,T^*(g)\big\rangle,$$

has a kernel that coincides with the function $K^*(x) = \overline{K(-x)}$ on $\mathbf{R}^n \setminus \{0\}$. We notice that since K satisfies (5.3.12), then so does K^* and with the same bound. Therefore, T^*, which maps $L^{r'}$ to $L^{r'}$, has a kernel that satisfies (5.3.12). By the preceding argument, T^* maps $L^{p'}$ to $L^{p'}$ with bound at most $C_n'(A_2+B)(p'-1)^{-1/p'}$ whenever $1 < p' < r'$. By duality this yields that T maps $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ with bound at most $C_n'(A_2+B)(p-1)^{1-1/p}$ whenever $r < p < \infty$. Using interpolation we obtain that T maps L^p to itself with norm at most $C_n'(A_2+B)\max((p-1)^{-1/p},(p-1)^{1-1/p})$ for $p > r$ and $p < r$. ~~in the interval~~ $\cancel{(r,r')}$ ~~, which is nonempty only if~~ $\cancel{r<2}$. Then (5.3.14) holds since $\max((p-1)^{-1/p},(p-1)^{1-1/p}) \le \max((p-1)^{-1},p)$. □

5.3.4 Discussion on Maximal Singular Integrals

In this subsection we introduce maximal singular integrals and we derive their boundedness under certain smoothness conditions on the kernels, assuming boundedness of the associated linear operator.

Suppose that K is a kernel on $\mathbf{R}^n \setminus \{0\}$ that satisfies the size condition

$$|K(x)| \le A_1|x|^{-n} \tag{5.3.15}$$

for $x \neq 0$. Then for any $\varepsilon > 0$ the function $K^{(\varepsilon)}(x) = K(x)\chi_{|x|\ge\varepsilon}$ lies in $L^{p'}(\mathbf{R}^n)$ (with norm bounded by $c_{p,n}A_1\varepsilon^{-n/p}$) for all $1 \le p < \infty$. Consequently, by Hölder's inequality, the integral

$$(f * K^{(\varepsilon)})(x) = \int_{|y|\ge\varepsilon} f(x-y)K(y)\,dy$$

converges absolutely for all $x \in \mathbf{R}^n$ and all $f \in L^p(\mathbf{R}^n)$, when $1 \le p < \infty$.