$$\big\|T\big\|_{L^1\to L^{1,\infty}} \le C_n\,(A_2+B), \tag{5.3.13}$$

and T also extends to a bounded operator from $L^p(\mathbf{R}^n)$ to itself for $1<p<\infty$ with norm

$$\big\|T\big\|_{L^p\to L^p} \le C_n' \max\big(p,(p-1)^{-1}\big)\,(A_2+B), \tag{5.3.14}$$

where C_n, C_n' are constants that depend on the dimension but not on r or p.

Proof. We discuss the case $r<\infty$ and we refer to Exercise 5.3.7 for the case $r=\infty$. Let $\alpha>0$ be given. We fix a step function f given as a finite linear combination of characteristic functions of disjoint dyadic cubes. The class of such functions is dense in all the L^p spaces. Once (5.3.13) is obtained for such functions, a density argument gives that T admits an extension on L^1 that also satisfies (5.3.13). Therefore it suffices to prove (5.3.13) for such a function f.

Apply the Calderón–Zygmund decomposition to f at height $\gamma\alpha$, where γ is a positive constant to be chosen later. That is, write the function f as the sum

$$f = g+b = g+\sum_j b_j,$$

where conditions (1)–(6) of Theorem 5.3.1 are satisfied with the constant α replaced by $\gamma\alpha$. Since f is a finite linear combination of characteristic functions of disjoint dyadic cubes, there are only finitely many cubes Q_j that appear in the Calderón–Zygmund decomposition to f. Each b_j is supported in a dyadic cube Q_j with center y_j and the Q_j's are pairwise disjoint. We denote by $\ell(Q)$ the side length of a cube Q. Let Q_j^* be the unique cube with sides parallel to the axes having the same center as Q_j and having side length $\ell(Q_j^*)=2\sqrt{n}\,\ell(Q_j)$. Because of the form of f, each b_j is a bounded function supported in $\overline{Q_j}$, hence it is in L^r, thus each $T(b_j)$ is a well-defined L^r function. We observe that for all j and all $x\notin Q_j^*$ we have

$$T(b_j)(x) = \lim_{k\to\infty}\int_{k\ge|x-y|\ge\delta_k} K(x-y)b_j(y)\,dy = \int_{Q_j} K(x-y)b_j(y)\,dy\,,$$

where the last integral converges absolutely. This is a consequence of the Lebesgue dominated convergence theorem, based on the facts that b_j is bounded, that K is integrable over any compact annulus that does not contain the origin (cf. (5.3.4)), and that $x-Q_j$ is contained in such a compact annulus, since $x\notin Q_j^*$.

Next we use the cancellation of b_j in the following way:

$$\begin{aligned}
&\int_{(\cup_i Q_i^*)^c}\sum_j \big|T(b_j)(x)\big|\,dx\\
&\qquad=\int_{(\bigcup_i Q_i^*)^c}\sum_j\bigg|\int_{Q_j} b_j(y)\big(K(x-y)-K(x-y_j)\big)\,dy\bigg|\,dx\\
&\qquad\le\sum_j\int_{(Q_j^*)^c}\int_{Q_j}|b_j(y)||K(x-y)-K(x-y_j)|\,dy\,dx
\end{aligned}$$