5.3.2 General Singular Integrals

The kernels of the general singular integrals we will study are tempered distributions that coincide with functions away from the origin. The setup is as follows. Let K be a measurable function defined on $\mathbf{R}^n \setminus \{0\}$ that is integrable on compact subsets of $\mathbf{R}^n \setminus \{0\}$ and satisfies the size condition

$$\sup_{R>0} \int_{R\le|x|\le 2R} |K(x)|\,dx = A_1 < \infty. \tag{5.3.4}$$

This condition is less restrictive than the standard size estimate

$$\sup_{x\in\mathbf{R}^n} |x|^n |K(x)| < \infty, \tag{5.3.5}$$

but it is strong enough to capture size properties of kernels $K(x) = \Omega(x/|x|)/|x|^n$, where $\Omega \in L^1(\mathbf{S}^{n-1})$. We also note that condition (5.3.4) is equivalent to

$$\sup_{R>0} \frac{1}{R} \int_{|x|\le R} |K(x)|\,|x|\,dx < \infty. \tag{5.3.6}$$

See Exercise 5.3.1.

The size condition (5.3.4) is sufficient to make the restriction of $K(x)$ on $|x| > \delta$ a tempered distribution (for any $\delta > 0$). Indeed, for $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we have

$$\begin{aligned}
\int_{|x|\ge 1} |K(x)\varphi(x)|\,dx \;&\le\; \sum_{m=0}^{\infty} \int_{2^{m+1}\ge|x|\ge 2^m} \frac{|K(x)|(1+|x|)^N|\varphi(x)|}{(1+2^m)^N}\,dx \\
&\le\; \sum_{m=0}^{\infty} \frac{A_1}{(1+2^m)^N} \sup_{x\in\mathbf{R}^n} (1+|x|)^N |\varphi(x)|\,,
\end{aligned}$$

and this expression is bounded by a constant times a finite sum of Schwartz seminorms of φ.

We are interested in tempered distributions W on $\mathbf{R}^n$ that extend the function K defined on $\mathbf{R}^n \setminus \{0\}$ and have the form

$$\langle W, \varphi\rangle = \lim_{j\to\infty} \int_{|x|\ge\delta_j} K(x)\varphi(x)\,dx, \qquad \varphi \in \mathscr{S}(\mathbf{R}^n), \tag{5.3.7}$$

for some sequence $\delta_j \downarrow 0$ as $j \to \infty$. It is not hard to see that there exists a tempered distribution W satisfying (5.3.7) for all $\varphi \in \mathscr{S}(\mathbf{R}^n)$ if and only if

$$\lim_{j\to\infty} \int_{1\ge|x|\ge\delta_j} K(x)\,dx = L \tag{5.3.8}$$

exists. See Exercise 5.3.2. If such a distribution W exists it may not be unique, since it depends on the choice of the sequence δ_j. Two different sequences tending to zero