Remark 5.2.9. It follows from the proof of Theorem 5.2.7 and from Theorems 5.1.7 and 5.1.12 that whenever Ω is an odd function on $\mathbf{S}^{n-1}$, we have

$$\begin{aligned}\big\|T_\Omega\big\|_{L^p\to L^p} &\le \big\|\Omega\big\|_{L^1}\begin{cases} a\,p & \text{when } p\ge 2,\\ a\,(p-1)^{-1} & \text{when } 1<p\le 2,\end{cases}\\ \big\|T_\Omega^{(**)}\big\|_{L^p\to L^p} &\le \big\|\Omega\big\|_{L^1}\begin{cases} a\,p & \text{when } p\ge 2,\\ a\,(p-1)^{-2} & \text{when } 1<p\le 2,\end{cases}\end{aligned}$$

for some $a>0$ independent of p and the dimension.

5.2.4 Singular Integrals with Even Kernels

Since a general integrable function Ω on $\mathbf{S}^{n-1}$ with mean value zero can be written as a sum of an odd and an even function, it suffices to study singular integral operators T_Ω with even kernels. For the rest of this section, fix an integrable even function Ω on $\mathbf{S}^{n-1}$ with mean value zero. The following idea is fundamental in the study of such singular integrals. Proposition 5.1.16 implies that

$$T_\Omega = -\sum_{j=1}^n R_j R_j T_\Omega\,. \tag{5.2.23}$$

If R_jT_Ω were another singular integral operator of the form T_{Ω_j} for some odd Ω_j, then the boundedness of T_Ω would follow from that of T_{Ω_j} via the identity (5.2.23) and Theorem 5.2.7. It turns out that R_jT_Ω does have an odd kernel, but it may not be integrable on $\mathbf{S}^{n-1}$ unless Ω itself possesses an additional amount of integrability. The amount of extra integrability needed is logarithmic, more precisely of this sort:

$$c_\Omega = \int_{\mathbf{S}^{n-1}} |\Omega(\theta)|\log(e+|\Omega(\theta)|)\,d\theta < \infty. \tag{5.2.24}$$

Observe that we always have

$$\big\|\Omega\big\|_{L^1} \le c_\Omega\,.$$

The following theorem is the main result of this section.

Theorem 5.2.10. *Let $n\ge 2$ and let Ω be an even integrable function on $\mathbf{S}^{n-1}$ with mean value zero that satisfies (5.2.24). Then the corresponding singular integral T_Ω is bounded on $L^p(\mathbf{R}^n)$, $1<p<\infty$, with norm at most a dimensional constant multiple of the quantity* $\max\big((p-1)^{-2},p^2\big)\,(c_\Omega+1)$.

If the operator T_Ω in Theorem 5.2.10 is weak type $(1,1)$, then the estimate on the L^p operator norm of T_Ω can be improved to $\|T_\Omega\|_{L^p\to L^p}\le C_n(p-1)^{-1}(c_\Omega+1)$ as $p\to 1$. This is indeed the case; see the historical comments at the end of this chapter.

Proof. Let W_Ω be the distributional kernel of T_Ω. We have that W_Ω coincides with the function $\Omega(x/|x|)|x|^{-n}$ on $\mathbf{R}^n\setminus\{0\}$. Using Proposition 5.2.3 and the fact that Ω is an even function, we obtain the formula

$$\widehat{W_\Omega}(\xi)=\int_{\mathbf{S}^{n-1}}\Omega(\theta)\log\frac{1}{|\xi\cdot\theta|}\,d\theta\,, \tag{5.2.25}$$

which implies that $\widehat{W_\Omega}$ is itself an even function. Now, using Exercise 5.2.3 and condition (5.2.24), we conclude that $\widehat{W_\Omega}$ is a bounded function. Therefore, T_Ω is L^2 bounded. To obtain the L^p boundedness of T_Ω, we use the idea mentioned earlier involving the Riesz transforms. In view of (5.1.46), we have that

$$T_\Omega=-\sum_{j=1}^{n}R_jT_j, \tag{5.2.26}$$

where $T_j=R_jT_\Omega$. Equality (5.2.26) makes sense as an operator identity on $L^2(\mathbf{R}^n)$, since T_Ω and each R_j are well defined and bounded on $L^2(\mathbf{R}^n)$.

The kernel of the operator T_j is the inverse Fourier transform of the distribution $-i\frac{\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)$, which we denote by K_j. At this point we know only that K_j is a tempered distribution whose Fourier transform is the function $-i\frac{\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)$. Our first goal is to show that K_j coincides with an integrable function on an annulus. To prove this assertion we write

$$W_\Omega=W_\Omega^0+W_\Omega^1+W_\Omega^\infty\,,$$

where W_Ω^0 is a distribution and $W_\Omega^1,W_\Omega^\infty$ are functions defined by

$$\begin{aligned}\left\langle W_\Omega^0,\varphi\right\rangle &= \lim_{\varepsilon\to0}\int_{\varepsilon<|x|\le\frac12}\frac{\Omega(x/|x|)}{|x|^n}\varphi(x)\,dx\,,\\ W_\Omega^1(x) &= \frac{\Omega(x/|x|)}{|x|^n}\chi_{\frac12\le|x|\le2}\,,\\ W_\Omega^\infty(x) &= \frac{\Omega(x/|x|)}{|x|^n}\chi_{2<|x|}\,.\end{aligned}$$

We now fix a $j\in\{1,2,\dots,n\}$ and we write

$$K_j=K_j^0+K_j^1+K_j^\infty\,,$$

where

$$\begin{aligned}K_j^0 &= \big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^0}(\xi)\big)^\vee\,,\\ K_j^1 &= \big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^1}(\xi)\big)^\vee\,,\\ K_j^\infty &= \big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^\infty}(\xi)\big)^\vee\,.\end{aligned}$$

Notice that K_j^0 is well defined via Theorem 2.3.21.

Define the annulus

$$A = \{x \in \mathbf{R}^n :\ 2/3 < |x| < 3/2\}.$$

For a smooth function ϕ supported in the annulus $2/3 < |x| < 3/2$ we have

$$\begin{aligned}
\langle K_j^0, \phi\rangle &= \big\langle \big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^0}(\xi)\big)^\vee, \phi\big\rangle \\
&= \big\langle -i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^0}(\xi), \phi^\vee(\xi)\big\rangle \\
&= \big\langle \widehat{W_\Omega^0}(\xi), -i\tfrac{\xi_j}{|\xi|}\phi^\vee(\xi)\big\rangle \\
&= \big\langle W_\Omega^0, \big(-i\tfrac{\xi_j}{|\xi|}\phi^\vee(\xi)\big)^\wedge\big\rangle \\
&= -\big\langle W_\Omega^0, \widetilde{R_j(\phi)}\big\rangle \\
&= -\lim_{\varepsilon\to 0}\int\limits_{\varepsilon<|y|<1/2} \frac{\Omega(y/|y|)}{|y|^n} R_j(\phi)(-y)\,dy \qquad (\Omega \text{ is even}) \\
&= -\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\lim_{\varepsilon\to 0}\int\limits_{\varepsilon<|y|<1/2}\frac{\Omega(y/|y|)}{|y|^n}\int\limits_{\mathbf{R}^n}\frac{y_j-x_j}{|y-x|^{n+1}}\phi(x)\,dx\,dy,
\end{aligned}$$

noticing that $|y-x|$ stays away from zero when $|y| < 1/2$ and x lies in A. It should be noted that the function $\zeta = \big(-i\frac{\xi_j}{|\xi|}\phi^\vee(\xi)\big)^\wedge$ is not Schwartz, but it is Lipschitz, i.e., it satisfies $|\zeta(x)-\zeta(y)| \le C_\phi|x-y|$, and the compactly supported tempered distribution W_Ω^0 can be extended to act on such functions.

It follows that K_j^0 coincides in A with the function inside the absolute value below:

$$\left|\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<\frac12}\frac{x_j-y_j}{|x-y|^{n+1}}\frac{\Omega(y/|y|)}{|y|^n}\,dy\right| \tag{5.2.27}$$

$$\begin{aligned}
&= \left|\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{|y|<\frac12}\left(\frac{x_j-y_j}{|x-y|^{n+1}}-\frac{x_j}{|x|^{n+1}}\right)\frac{\Omega(y/|y|)}{|y|^n}\,dy\right| \qquad (5.2.28) \\
&\le \int_{|y|\le\frac12} C_n|y|\,\frac{|\Omega(y/|y|)|}{|y|^n}\,dy \\
&= C_n'\big\|\Omega\big\|_{L^1},
\end{aligned}$$

where we used the fact that $\Omega(y/|y|)|y|^{-n}$ has integral zero over annuli of the form $\varepsilon < |y| < \frac12$, the mean value theorem applied to the function $x_j|x|^{-(n+1)}$, and the fact that $|x-y| \ge 1/6$ for x in the annulus A. We conclude that on A, K_j^0 coincides with the bounded function inside the absolute value in (5.2.27).

Likewise, for $x \in A$ we have

$$\begin{aligned}
&\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\left|\int_{|y|>2}\frac{x_j-y_j}{|x-y|^{n+1}}\frac{\Omega(y/|y|)}{|y|^n}\,dy\right| \qquad (5.2.29)\\
&\le \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{|y|>2}\frac{1}{|x-y|^n}\frac{|\Omega(y/|y|)|}{|y|^n}\,dy\\
&\le \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{|y|>2}\frac{4^n}{|y|^{2n}}|\Omega(y/|y|)|\,dy\\
&= C\big\|\Omega\big\|_{L^1},
\end{aligned}$$

from which it follows that on the annulus A, K_j^∞ coincides with the bounded function inside the absolute value in (5.2.29) or in (5.2.28).

Now observe that condition (5.2.24) gives that the function W_Ω^1 satisfies

$$\begin{aligned}
&\int_{|x|\le 2}|W_\Omega^1(x)|\log(e+|W_\Omega^1(x)|)\,dx\\
&\qquad \le \int_{1/2}^2\int_{\mathbf{S}^{n-1}}\frac{|\Omega(\theta)|}{r^n}\log\big[e+2^n|\Omega(\theta)|\big]\,d\theta\, r^{n-1}\,dr\\
&\qquad \le (\log 4)\,\big[n(\log 2)\big\|\Omega\big\|_{L^1}+c_\Omega\big] \le c_n' c_\Omega < \infty.
\end{aligned}$$

Since the Riesz transform R_j is countably subadditive and maps L^p to L^p with norm at most $4(p-1)^{-1}$ for $1<p<2$, it follows from Exercise 1.3.7 that $K_j^1 = R_j(W_\Omega^1)$ is integrable over the ball $|x|\le 3/2$ and moreover, it satisfies

$$\int_A |K_j^1(x)|\,dx \le C_n\left[\int_{|x|\le 2}|W_\Omega^1(x)|\log^+|W_\Omega^1(x)|\,dx+1\right] \le C_n'\,(c_\Omega+1)\,.$$

Furthermore, since $\widehat{K_j}$ is homogeneous of degree zero, K_j is a homogeneous distribution of degree $-n$ (Exercise 2.3.9). This means that for all test functions φ and all $\lambda>0$ we have

$$\big\langle K_j, \delta^\lambda(\varphi)\big\rangle = \big\langle K_j,\varphi\big\rangle\,, \qquad (5.2.30)$$

where $\delta^\lambda(\varphi)(x)=\varphi(\lambda x)$. But for $\varphi\in\mathscr{C}_0^\infty$ supported in the annulus $3/4<|x|<4/3$ and for λ in $(8/9,9/8)$ we have that $\delta^{\lambda^{-1}}(\varphi)$ is supported in A and thus we can express (5.2.30) as convergent integrals as follows:

$$\int_{\mathbf{R}^n}K_j(x)\varphi(x)\,dx = \int_{\mathbf{R}^n}K_j(x)\varphi(\lambda^{-1}x)\,dx = \int_{\mathbf{R}^n}\lambda^n K_j(\lambda x)\varphi(x)\,dx\,. \qquad (5.2.31)$$

From this it would be ideal to be able to directly obtain that $K_j(x)=\lambda^n K_j(\lambda x)$ for all $8/9<|x|<9/8$ and $8/9<\lambda<9/8$, in particular when $\lambda=|x|^{-1}$. But unfortunately, we can only deduce that for every $\lambda\in(8/9,9/8)$, $K_j(x)=\lambda^n K_j(\lambda x)$ holds for all x in the annulus except a set of measure zero that depends on λ. To be able to define the restriction of K_j on $\mathbf{S}^{n-1}$, we employ a more delicate argument.

For any J subinterval of $[8/9,9/8]$ we obtain from (5.2.31) that

$$\int_{\mathbf{R}^n} K_j(x)\varphi(x)\,dx = \int_{\mathbf{R}^n} \fint_J \lambda^n K_j(\lambda x)\,d\lambda\,\varphi(x)\,dx\,,$$

where integral with the slashed integral denotes the average of a function over the set J. Since φ was an arbitrary $\mathscr{C}_0^\infty$ function supported in the annulus $3/4<|x|<4/3$, it follows that for every J subinterval of $[8/9,9/8]$, there is a null[1] subset E_J of the annulus $A'=\{x:\ 27/32<|x|<32/27\}$ such that

$$K_j(x) = \fint_J \lambda^n K_j(\lambda x)\,d\lambda \tag{5.2.32}$$

for all $x\in A'\setminus E_J$.

Let $J_0=[\sqrt{8/9},\sqrt{9/8}]$. We claim that there is a null subset E of A' such that for all $x\in A'\setminus E$ we have

$$\fint_{J_0} \lambda^n K_j(\lambda x)\,d\lambda = \fint_{rJ_0} \lambda^n K_j(\lambda x)\,d\lambda \tag{5.2.33}$$

for every r in J_0. Indeed, let E be the union of E_{rJ_0} over all r in $J_0\cap\mathbf{Q}$. Then in view of (5.2.32), identity (5.2.33) holds for $x\in A'\setminus E$ and $J_0\cap\mathbf{Q}$. But for a fixed x in $A'\setminus E$, the function of r on the right hand side of (5.2.33) is constant on the rationals and is also continuous (in r), hence it must be constant for all $r\in J_0$. Thus the claim follows since both sides of (5.2.33) are equal to (5.2.32).

Writing $x=\delta\theta$, where $27/32<\delta<32/27$ and $\theta\in\mathbf{S}^{n-1}$, it follows by Fubini's theorem that there is a $\delta\in(27/32,32/27)$ (in fact almost all δ have this property) such that

$$\fint_{J_0} \lambda^n K_j(\lambda\delta\theta)\,d\lambda = \fint_{rJ_0} \lambda^n K_j(\lambda\delta\theta)\,d\lambda \tag{5.2.34}$$

for almost all $\theta\in\mathbf{S}^{n-1}$ and all $r\in J_0$. We fix such a δ, which we denote δ_0.

We now define a function Ω_j on $\mathbf{S}^{n-1}$ by setting

$$\Omega_j(\theta) = \fint_{J_0} \delta_0^n\lambda^n K_j(\lambda\delta_0\theta)\,d\lambda = \fint_{rJ_0} \delta_0^n\lambda^n K_j(\lambda\delta_0\theta)\,d\lambda$$

for all $r\in J_0$. The function Ω_j is defined almost everywhere and is integrable over $\mathbf{S}^{n-1}$, since K_j is integrable over the annulus A.

Let $e_1=(1,0,\dots,0)$. Let Ψ be a $\mathscr{C}_0^\infty(\mathbf{R}^n)$ nonzero, nonnegative, radial, and supported in the annulus $32/(27\sqrt{2})<|x|<27\sqrt{2}/32$ around $\mathbf{S}^{n-1}$. We start with

$$\Omega_j(\theta) = \fint_{r^{-1}J_0} \delta_0^n\lambda^n K_j(\lambda\delta_0\theta)\,d\lambda = \fint_{J_0} \delta_0^n r^n\lambda^n K_j(r\lambda\delta_0\theta)\,d\lambda\,,$$

which holds for all $r\in J_0$, we multiply by $\Psi(re_1)$, and we integrate over $\mathbf{S}^{n-1}$ and over $(0,\infty)$ with respect to the measure dr/r. We obtain

---

[1] here we are making use of the following version of du Bois-Reymond's lemma: if U is an open subset of $\mathbf{R}^n$ and g is an integrable function on U such that $\int_U g(x)\psi(x)dx=0$ for all ψ smooth functions with compact support contained in U, then $g=0$ a.e. on U.

$$\begin{aligned}
\int_0^\infty \Psi(re_1)\frac{dr}{r}\int_{\mathbf{S}^{n-1}}\Omega_j(\theta)\,d\theta &= \fint_{J_0}\int_0^\infty\int_{\mathbf{S}^{n-1}}\delta_0^n\lambda^n K_j(\lambda\delta_0 r\theta)\Psi(re_1)r^n d\theta\frac{dr}{r}d\lambda \\
&= \fint_{J_0}\int_{\mathbf{R}^n}\delta_0^n\lambda^n K_j(\lambda\delta_0 x)\,\Psi(x)dxd\lambda \\
&= \fint_{J_0}\int_{\mathbf{R}^n}K_j(x)\Psi((\lambda\delta_0)^{-1}x)\,dx\,d\lambda \\
&= \fint_{J_0}\big\langle K_j,\Psi\big\rangle\,d\lambda\,, \\
&= \big\langle K_j,\Psi\big\rangle
\end{aligned}$$

in view of the homogeneity of K_j. But, as $\widehat{\Psi}=\Psi^\vee$, for some constant c'_Ψ we have

$$\big\langle K_j,\Psi\big\rangle = \big\langle \widehat{K_j},\widehat{\Psi}^\vee\big\rangle = \int_{\mathbf{R}^n}\frac{-i\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)\widehat{\Psi}(\xi)\,d\xi = c'_\Psi\int_{\mathbf{S}^{n-1}}\frac{-i\theta_j}{|\theta|}\widehat{W_\Omega}(\theta)\,d\theta = 0,$$

since by (5.2.25), $\frac{-i\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)$ is an odd function. We conclude that Ω_j has mean value zero over $\mathbf{S}^{n-1}$.

Thus $\Omega_j\in L^1(\mathbf{S}^{n-1})$ has mean value zero and the distribution W_{Ω_j} is well defined. We claim that

$$K_j = W_{\Omega_j}\,. \tag{5.2.35}$$

To establish (5.2.35), we show first that $\big\langle K_j,\varphi\big\rangle = \big\langle W_{\Omega_j},\varphi\big\rangle$ whenever φ is supported in the annulus $8/9<|x|<9/8$. Using (5.2.32) we have

$$\begin{aligned}
\int_{\mathbf{R}^n}K_j(x)\varphi(x)\,dx &= \int_{\mathbf{R}^n}\fint_{J_0}K_j(\delta_0\lambda x)\delta_0^n\lambda^n d\lambda\,\varphi(x)\,dx \\
&= \int_0^\infty\int_{\mathbf{S}^{n-1}}\fint_{J_0}K_j(\delta_0\lambda r\theta)\delta_0^n\lambda^n r^n\,d\lambda\,\varphi(r\theta)\,d\theta\frac{dr}{r} \\
&= \int_0^\infty\int_{\mathbf{S}^{n-1}}\fint_{rJ_0}K_j(\delta_0\lambda'\theta)\delta_0^n(\lambda')^n d\lambda'\,\varphi(r\theta)\,d\theta\frac{dr}{r} \\
&= \int_0^\infty\int_{\mathbf{S}^{n-1}}\Omega_j(\theta)\varphi(r\theta)\,d\theta\frac{dr}{r} \\
&= \big\langle W_{\Omega_j},\varphi\big\rangle,
\end{aligned}$$

having used (5.2.34) in the second to last equality.

Given a general $\mathscr{C}_0^\infty$ function φ whose support is contained in an annulus of the form $M^{-1}<|x|<M$, for some $M>0$, via a smooth partition of unity, we write φ as a finite sum of smooth functions φ_k whose supports are contained in annuli of the form $8s/9<|x|<9s/8$ for some $s>0$. These annuli can be brought inside the annulus $8/9<|x|<9/8$ by a dilation. Since both K_j and W_{Ω_j} are homogeneous distributions of degree $-n$ and agree on the annulus $8/9<|x|<9/8$ they must agree on annuli $8s/9<|x|<9s/8$. Consequently, $\langle K_j,\varphi\rangle = \langle W_{\Omega_j},\varphi\rangle$ for all $\varphi\in\mathscr{C}_0^\infty(\mathbf{R}^n\setminus\{0\})$. Therefore, $K_j-W_{\Omega_j}$ is supported at the origin, and since it is homogeneous of degree $-n$, it must be equal to $b\delta_0$, a constant multiple of the Dirac mass. But $\widehat{K_j}$ is an odd function and hence K_j is also odd. It follows that W_{Ω_j} is an odd function on $\mathbf{R}^n \setminus \{0\}$, which implies that Ω_j is an odd function. We say that $u \in \mathscr{S}'(\mathbf{R}^n)$ is odd if $\widetilde{u} = -u$, where $\widetilde{u}$ is defined by $\langle \widetilde{u}, \psi \rangle = \langle u, \widetilde{\psi} \rangle$ for all $\psi \in \mathscr{S}(\mathbf{R}^n)$ and $\widetilde{\psi}(x) = \psi(-x)$. We have that $K_j - W_{\Omega_j}$ is an odd distribution, and thus $b\delta_0$ must be an odd distribution. But if $b\delta_0$ is odd, then $b = 0$. We conclude that for each j there exists an odd integrable function Ω_j on $\mathbf{S}^{n-1}$ with $\|\Omega_j\|_{L^1}$ controlled by a constant multiple of c_Ω such that (5.2.35) holds.

Then we use (5.2.26) and (5.2.35) to write

$$T_\Omega = -\sum_{j=1}^{n} R_j T_{\Omega_j},$$

and appealing to the boundedness of each T_{Ω_j} (Theorem 5.2.7) and to that of the Riesz transforms, we obtain the required L^p boundedness for T_Ω. □

We note that Theorem 5.2.10 holds for all $\Omega \in L^1(\mathbf{S}^{n-1})$ that satisfy (5.2.24), not necessarily even Ω. Simply write $\Omega = \Omega_e + \Omega_o$, where Ω_e is even and Ω_o is odd, and check that condition (5.2.24) holds for Ω_e.

5.2.5 Maximal Singular Integrals with Even Kernels

We have the corresponding theorem for maximal singular integrals.

Theorem 5.2.11. *Let $n \geq 2$ and let Ω be an even integrable function on $\mathbf{S}^{n-1}$ with mean value zero that satisfies (5.2.24). Then the corresponding maximal singular integral $T_\Omega^{(**)}$, defined in* (5.2.4)*, is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ with norm at most a dimensional constant multiple of* $\max(p^2, (p-1)^{-3})\,(c_\Omega + 1)$.

Proof. For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, we define the maximal function of f in the direction θ by setting

$$M_\theta(f)(x) = \sup_{a>0} \frac{1}{2a} \int_{|r|\leq a} |f(x - r\theta)|\, dr. \tag{5.2.36}$$

In view of Exercise 5.2.5 we have that M_θ is bounded on $L^p(\mathbf{R}^n)$ with norm at most $3\,p\,(p-1)^{-1}$.

Fix Φ a smooth radial function such that $\Phi(x) = 0$ for $|x| \leq 1/4$, $\Phi(x) = 1$ for $|x| \geq 3/4$, and $0 \leq \Phi(x) \leq 1$ for all x in $\mathbf{R}^n$. For $f \in L^p(\mathbf{R}^n)$ and $0 < \varepsilon < N < \infty$ we introduce the smoothly truncated singular integral

$$\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) = \int_{\mathbf{R}^n} \frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n} \Big(\Phi\big(\tfrac{y}{\varepsilon}\big) - \Phi\big(\tfrac{y}{N}\big) \Big) f(x-y)\, dy$$

and the corresponding maximal singular integral operator

$$\widetilde{T}_\Omega^{(**)}(f) = \sup_{0<N<\infty} \sup_{0<\varepsilon<N} |\widetilde{T}_\Omega^{(\varepsilon,N)}(f)|. \tag{5.2.37}$$

It suffices to work with $\widetilde{T}_\Omega^{(**)}$ instead of $T_\Omega^{(**)}$ in view of the following argument.

For f in $L^p(\mathbf{R}^n)$ (for some $1<p<\infty$), we have

$$
\begin{aligned}
&\Big| |\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x)| - |T_\Omega^{(\varepsilon,N)}(f)(x)| \Big| \\
&\quad \le \big|\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) - T_\Omega^{(\varepsilon,N)}(f)(x)\big| \\
&\quad = \bigg| \bigg[\int\limits_{|y|\ge\frac{\varepsilon}{4}} \frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n} \Phi\big(\tfrac{y}{\varepsilon}\big) f(x-y)\,dy - \int\limits_{|y|\ge\frac{N}{4}} \frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n} \Phi\big(\tfrac{y}{N}\big) f(x-y)\,dy \bigg] \\
&\qquad - \bigg[\int\limits_{|y|\ge\varepsilon} \frac{|\Omega\big(\frac{y}{|y|}\big)|}{|y|^n} \Phi\big(\tfrac{y}{\varepsilon}\big) f(x-y)\,dy - \int\limits_{|y|\ge N} \frac{|\Omega\big(\frac{y}{|y|}\big)|}{|y|^n} \Phi\big(\tfrac{y}{N}\big) f(x-y)\,dy \bigg] \bigg| \\
&\quad \le \int\limits_{\mathbf{S}^{n-1}} |\Omega(\theta)| \bigg[\frac{4}{\varepsilon} \int_{\frac{\varepsilon}{4}}^{\varepsilon} |f(x-r\theta)|\,dr + \frac{4}{N} \int_{\frac{N}{4}}^{N} |f(x-r\theta)|\,dr \bigg] d\theta \\
&\quad \le 16 \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| M_\theta(f)(x)\,d\theta\,.
\end{aligned}
$$

Taking the supremum over $N>\varepsilon>0$ and using the result of Exercise 5.2.5 we conclude that

$$\big\|\widetilde{T}_\Omega^{(**)}(f) - T_\Omega^{(**)}(f)\big\|_{L^p} \le 100 \big\|\Omega\big\|_{L^1} \max(1,(p-1)^{-1}) \big\|f\big\|_{L^p}\,.$$

This implies that it suffices to obtain the required L^p bound for the smoothly truncated maximal singular integral operator $\widetilde{T}_\Omega^{(**)}$.

In proving the required estimate, we may assume that the even function Ω is bounded. For, if we know that for Ω even and bounded we had

$$\big\|\widetilde{T}_\Omega^{(**)}\big\|_{L^p\to L^p} \le C_n \max(p^2,(p-1)^{-3})(c_\Omega+1),$$

then given a general even function Ω in $L\log L(\mathbf{S}^{n-1})$ we set $\Omega^m = \Omega\chi_{|\Omega|\le m} - \kappa^m$, where κ^m are chosen so that $\int_{\mathbf{S}^{n-1}} \Omega^m\,d\sigma = 0$ for all $m\ge 1$. Then for all $x\in\mathbf{R}^n$

$$\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) \le \liminf_{m\to\infty} \widetilde{T}_{\Omega^m}^{(\varepsilon,N)}(f)(x) \le \liminf_{m\to\infty} \widetilde{T}_{\Omega^m}^{(**)}(f)(x)$$

whenever $f\in L^p\cap L^\infty$. Taking the supremum over $\varepsilon,N>0$ and applying Fatou's lemma and a density argument (passing from $L^p\cap L^\infty$ to L^p) we obtain

$$\big\|\widetilde{T}_\Omega^{(**)}\big\|_{L^p\to L^p} \le \liminf_{m\to\infty} \big\|\widetilde{T}_{\Omega^m}^{(**)}\big\|_{L^p\to L^p} \le C_n C(p) \liminf_{m\to\infty}(c_{\Omega^m}+1) = C_n C(p)(c_\Omega+1),$$

where $C(p)=\max(p^2,(p-1)^{-3})$.

So we fix a bounded function Ω on $\mathbf{S}^{n-1}$ with integral zero. Let K_j, Ω_j, and T_j be as in the previous theorem, and let F_j be the Riesz transform of the function $\Omega(x/|x|)\Phi(x)|x|^{-n}$. Let $f\in L^p(\mathbf{R}^n)$. A calculation yields the identity

$$
\begin{aligned}
\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) &= \int_{\mathbf{R}^n} \bigg[\frac{1}{\varepsilon^n} \frac{\Omega(\frac{y}{\varepsilon}/|\frac{y}{\varepsilon}|)}{|\frac{y}{\varepsilon}|^n} \Phi(\tfrac{y}{\varepsilon}) - \frac{1}{N^n} \frac{\Omega(\frac{y}{N}/|\frac{y}{N}|)}{|\frac{y}{N}|^n} \Phi(\tfrac{y}{N}) \bigg] f(x-y)\,dy \\
&= -\Big(\sum_{j=1}^n \Big[\tfrac{1}{\varepsilon^n} F_j\big(\tfrac{\cdot}{\varepsilon}\big) - \tfrac{1}{N^n} F_j\big(\tfrac{\cdot}{N}\big) \Big] * R_j(f) \Big)(x)\,,
\end{aligned}
$$

where in the last step we used Proposition 5.1.16. Therefore we may write

$$\begin{aligned} -\widetilde{T}_{\Omega}^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \int_{\mathbf{R}^n} \Big[\tfrac{1}{\varepsilon^n} F_j\big(\tfrac{x-y}{\varepsilon}\big) - \tfrac{1}{N^n} F_j\big(\tfrac{x-y}{N}\big) \Big] R_j(f)(y)\,dy \\ &= A_1^{(\varepsilon,N)}(f)(x) + A_2^{(\varepsilon,N)}(f)(x) + A_3^{(\varepsilon,N)}(f)(x)\,, \end{aligned} \tag{5.2.38}$$

where

$$\begin{aligned} A_1^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \int_{|x-y|\le\varepsilon} F_j\big(\tfrac{x-y}{\varepsilon}\big) R_j(f)(y)\,dy \\ &\qquad\qquad - \sum_{j=1}^{n} \frac{1}{N^n} \int_{|x-y|\le N} F_j\big(\tfrac{x-y}{N}\big) R_j(f)(y)\,dy\,, \\ A_2^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \int_{\mathbf{R}^n} \Big[\tfrac{1}{\varepsilon^n} \chi_{|x-y|>\varepsilon} \big\{ F_j\big(\tfrac{x-y}{\varepsilon}\big) - K_j\big(\tfrac{x-y}{\varepsilon}\big)\big\} \\ &\qquad\qquad - \tfrac{1}{N^n} \chi_{|x-y|>N} \big\{ F_j\big(\tfrac{x-y}{N}\big) - K_j\big(\tfrac{x-y}{N}\big)\big\} \Big] R_j(f)(y)\,dy\,, \\ A_3^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \int_{\mathbf{R}^n} \Big[\tfrac{1}{\varepsilon^n} \chi_{|x-y|>\varepsilon} K_j\big(\tfrac{x-y}{\varepsilon}\big) - \tfrac{1}{N^n} \chi_{|x-y|>N} K_j\big(\tfrac{x-y}{N}\big) \Big] R_j(f)(y)\,dy\,. \end{aligned}$$

It follows from the definitions of F_j and K_j that

$$\begin{aligned} F_j(z) - K_j(z) &= \frac{\Gamma\big(\frac{n+1}{2}\big)}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon\to 0} \int_{\varepsilon\le|y|} \frac{\Omega(y/|y|)}{|y|^n} \big(\Phi(y)-1\big) \frac{z_j-y_j}{|z-y|^{n+1}}\,dy \\ &= \frac{\Gamma\big(\frac{n+1}{2}\big)}{\pi^{\frac{n+1}{2}}} \int_{|y|\le\frac{3}{4}} \frac{\Omega(y/|y|)}{|y|^n} \big(\Phi(y)-1\big) \left\{ \frac{z_j-y_j}{|z-y|^{n+1}} - \frac{z_j}{|z|^{n+1}} \right\} dy \end{aligned}$$

whenever $|z|\ge 1$. But using the mean value theorem, the last expression is easily seen to be bounded by

$$C_n \int_{|y|\le\frac{3}{4}} \frac{|\Omega(y/|y|)|}{|y|^n} \frac{|y|}{|z|^{n+1}}\,dy = C_n' \big\|\Omega\big\|_{L^1} |z|^{-(n+1)}\,,$$

whenever $|z|\ge 1$. Using this estimate, we obtain that the jth term in $A_2^{(\varepsilon,N)}(f)(x)$ is bounded by

$$C_n \frac{\|\Omega\|_{L^1}}{\varepsilon^n} \int\limits_{|x-y|>\varepsilon} \frac{|R_j(f)(y)|\,dy}{(|x-y|/\varepsilon)^{n+1}} \le C_n \frac{2\|\Omega\|_{L^1}}{2^{-n}\varepsilon^n} \int\limits_{\mathbf{R}^n} \frac{|R_j(f)(y)|\,dy}{\big(1+\frac{|x-y|}{\varepsilon}\big)^{n+1}}\,.$$

It follows that for functions f in L^p we have

$$\sup_{0<\varepsilon<N<\infty} |A_2^{(\varepsilon,N)}(f)| \le C_n \big\|\Omega\big\|_{L^1} M(R_j(f))\,,$$

in view of Theorem 2.1.10. (M here is the Hardy–Littlewood maximal operator.) By Theorem 2.1.6, M maps $L^p(\mathbf{R}^n)$ to itself with norm bounded by a dimensional constant multiple of $\max(1,(p-1)^{-1})$. Since by Remark 5.2.9 the norm $\|R_j\|_{L^p\to L^p}$ is controlled by a dimensional constant multiple of $\max(p,(p-1)^{-1})$, it follows that

$$\Big\|\sup_{0<\varepsilon<N<\infty}|A_2^{(\varepsilon,N)}(f)|\Big\|_{L^p}\le C_n\big\|\Omega\big\|_{L^1}\max(p,(p-1)^{-2})\big\|f\big\|_{L^p}. \tag{5.2.39}$$

Next, recall that in the proof of Theorem 5.2.10 we showed that

$$K_j(x)=\frac{\Omega_j(x/|x|)}{|x|^n},$$

where Ω_j are integrable functions on $\mathbf{S}^{n-1}$ that satisfy

$$\big\|\Omega_j\big\|_{L^1}\le C_n\,(c_\Omega+1). \tag{5.2.40}$$

Consequently, for functions f in $L^p(\mathbf{R}^n)$ we have

$$\sup_{0<\varepsilon<N<\infty}|A_3^{(\varepsilon,N)}(f)|\le 2\sum_{j=1}^{n}T_{\Omega_j}^{(**)}(R_j(f)),$$

and by Remark 5.2.9 this last expression has L^p norm at most a dimensional constant multiple of $\big\|\Omega_j\big\|_{L^1}\max(p,(p-1)^{-2})\big\|R_j(f)\big\|_{L^p}$. It follows that

$$\Big\|\sup_{0<\varepsilon<N<\infty}|A_3^{(\varepsilon,N)}(f)|\Big\|_{L^p}\le C_n\max(p^2,(p-1)^{-3})(c_\Omega+1)\,\big\|f\big\|_{L^p}. \tag{5.2.41}$$

Finally, we turn our attention to the term $A_1^{(\varepsilon,N)}(f)$. To prove the required estimate, we first show that there exist nonnegative homogeneous of degree zero functions G_j on $\mathbf{R}^n$ that satisfy

$$|F_j(x)|\le G_j(x)\qquad\text{when } |x|\le 1 \tag{5.2.42}$$

and

$$\int_{\mathbf{S}^{n-1}}|G_j(\theta)|\,d\theta\le C_n\,c_\Omega. \tag{5.2.43}$$

To prove (5.2.42), first note that if $|x|\le 1/8$, then

$$\begin{aligned}|F_j(x)|&=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\bigg|\int_{\mathbf{R}^n}\frac{\Omega(y/|y|)}{|y|^n}\Phi(y)\frac{x_j-y_j}{|x-y|^{n+1}}dy\bigg|\\&\le C_n\int_{|y|\ge\frac14}\frac{|\Omega(y/|y|)|}{|y|^{2n}}\,dy\\&\le C_n'\big\|\Omega\big\|_{L^1}.\end{aligned}$$

We now fix an x satisfying $1/8 \le |x| \le 1$ and we write

$$\begin{aligned} |F_j(x)| &\le \Phi(x)|K_j(x)| + |F_j(x) - \Phi(x)K_j(x)| \\ &\le |K_j(x)| + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \left| \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} \big(\Phi(y) - \Phi(x)\big) \frac{\Omega(y/|y|)}{|y|^n} dy \right| \\ &= |K_j(x)| + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \big(P_1(x) + P_2(x) + P_3(x)\big), \end{aligned}$$

where

$$\begin{aligned} P_1(x) &= \left| \int_{|y|\le\frac{1}{16}} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) \big(\Phi(y) - \Phi(x)\big) \frac{\Omega(y/|y|)}{|y|^n}\, dy \right|, \\ P_2(x) &= \left| \int_{\frac{1}{16}\le|y|\le 2} \frac{x_j - y_j}{|x-y|^{n+1}} \big(\Phi(y) - \Phi(x)\big) \frac{\Omega(y/|y|)}{|y|^n}\, dy \right|, \\ P_3(x) &= \left| \int_{|y|\ge 2} \frac{x_j - y_j}{|x-y|^{n+1}} \big(\Phi(y) - \Phi(x)\big) \frac{\Omega(y/|y|)}{|y|^n}\, dy \right|. \end{aligned}$$

But since $1/8 \le |x| \le 1$, we see that

$$P_1(x) \le C_n \int_{|y|\le\frac{1}{16}} \frac{|y|}{|x|^{n+1}} \frac{|\Omega(y/|y|)|}{|y|^n}\, dy \le C_n' \big\|\Omega\big\|_{L^1}$$

and that

$$P_3(x) \le C_n \int_{|y|\ge 2} \frac{|\Omega(y/|y|)|}{|y|^{2n}}\, dy \le C_n' \big\|\Omega\big\|_{L^1}.$$

For $P_2(x)$ we use the estimate $|\Phi(y) - \Phi(x)| \le C|x-y|$ to obtain

$$\begin{aligned} P_2(x) &\le \int_{\frac{1}{16}\le|y|\le 2} \frac{C}{|x-y|^{n-1}} \frac{|\Omega(y/|y|)|}{|y|^n}\, dy \\ &\le 4C \int_{\frac{1}{16}\le|y|\le 2} \frac{|\Omega(y/|y|)|}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}}\, dy \\ &\le 4C \int_{\mathbf{R}^n} \frac{|\Omega(y/|y|)|}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}}\, dy. \end{aligned}$$

Recall that $K_j(x) = \Omega_j(x/|x|)|x|^{-n}$. We now set

$$G_j(x) = C_n \left(\big\|\Omega\big\|_{L^1} + \left|\Omega_j\Big(\frac{x}{|x|}\Big)\right| + |x|^{n-\frac{3}{2}} \int_{\mathbf{R}^n} \frac{|\Omega(y/|y|)|\, dy}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}} \right) \tag{5.2.44}$$

and we observe that G_j is a homogeneous of degree zero function, it satisfies (5.2.42), and it is integrable over the annulus $\frac{1}{2} \le |x| \le 2$. To verify the last assertion, we split up the double integral

$$I = \int_{\frac{1}{2}\le |x|\le 2} \int_{\mathbf{R}^n} \frac{|\Omega(y/|y|)|\,dy}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}}\,dx$$

into the pieces $1/4 \le |y| \le 4$, $|y| > 4$, and $|y| < 1/4$. The part of I where $1/4 \le |y| \le 4$ is pointwise bounded by a constant multiple of

$$\int\limits_{\frac{1}{4}\le|y|\le 4} \Big|\Omega\Big(\frac{y}{|y|}\Big)\Big| \int\limits_{\frac{1}{2}\le|x|\le 2} \frac{dx}{|y-x|^{n-1}}\,dy \le \int\limits_{\frac{1}{4}\le|y|\le 4} \Big|\Omega\Big(\frac{y}{|y|}\Big)\Big| \int\limits_{|x-y|\le 6} \frac{dx}{|y-x|^{n-1}}\,dy\,,$$

which is pointwise controlled by a constant multiple of $\|\Omega\|_{L^1}$. In the part of I where $|y| > 4$ we use that $|x-y|^{-n+1} \le (|y|/2)^{-n+1}$ to obtain rapid decay in y and hence a bound by a constant multiple of $\|\Omega\|_{L^1}$. Finally, in the part of I where $|y| < 1/4$ we use that $|x-y|^{-n+1} \le (1/4)^{-n+1}$, and then we also obtain a similar bound. It follows from (5.2.44) and (5.2.40) that

$$\int_{\frac{1}{2}\le|x|\le 2} |G_j(x)|\,dx \le C_n\big(\|\Omega\|_{L^1} + \|\Omega_j\|_{L^1} + \|\Omega\|_{L^1}\big) \le C_n\, c_\Omega\,.$$

Since G_j is homogeneous of degree zero, we deduce (5.2.43).

To complete the proof, we argue as follows:

$$\begin{aligned}
&\sup_{0<\varepsilon<N<\infty} |A_1^{(\varepsilon,N)}(f)(x)| \\
&\qquad \le 2\sup_{\varepsilon>0} \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \int_{|z|\le\varepsilon} |F_j\Big(\frac{z}{\varepsilon}\Big)|\,|R_j(f)(x-z)|\,dz \\
&\qquad \le 2\sup_{\varepsilon>0} \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \int_{r=0}^{\varepsilon}\int_{\mathbf{S}^{n-1}} \Big|F_j\Big(\frac{r\theta}{\varepsilon}\Big)\Big|\,|R_j(f)(x-r\theta)|\,r^{n-1}\,d\theta\,dr \\
&\qquad \le 2\sum_{j=1}^{n} \int_{\mathbf{S}^{n-1}} |G_j(\theta)| \Big\{\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_{r=0}^{\varepsilon} |R_j(f)(x-r\theta)|\,r^{n-1}\,dr\Big\}\,d\theta \\
&\qquad \le 4\sum_{j=1}^{n} \int_{\mathbf{S}^{n-1}} |G_j(\theta)| M_\theta(R_j(f))(x)\,d\theta\,.
\end{aligned}$$

Using (5.2.43) together with the L^p boundedness of the Riesz transforms and of M_θ we obtain

$$\Big\|\sup_{0<\varepsilon<N<\infty} |A_1^{(\varepsilon,N)}(f)|\Big\|_{L^p} \le C_n \max(p^2, (p-1)^{-3})(c_\Omega + 1)\big\|f\big\|_{L^p}\,. \tag{5.2.45}$$

Combining (5.2.45), (5.2.39), and (5.2.41), we obtain the required conclusion. □

The following corollary is a consequence of Theorem 5.2.11.

Corollary 5.2.12. *Let $n \ge 2$ and Ω be as in Theorem 5.2.11. Then for $1 < p < \infty$ and f in $L^p(\mathbf{R}^n)$ the functions $T_\Omega^{(\varepsilon,N)}(f)$ converge to $T_\Omega(f)$ in L^p and almost everywhere as $\varepsilon \to 0$ and $N \to \infty$.*

Proof. The a.e. convergence is a consequence of Theorem 2.1.14. The L^p convergence is a consequence of the Lebesgue dominated convergence theorem since for $f \in L^p(\mathbf{R}^n)$ we have that $|T_\Omega^{(\varepsilon,N)}(f)| \le T_\Omega^{(**)}(f)$ and $T_\Omega^{(**)}(f)$ is in $L^p(\mathbf{R}^n)$. $\square$

Exercises

5.2.1. Show that the directional Hilbert transform $\mathscr{H}_\theta$ is given by convolution with the distribution w_θ in $\mathscr{S}'(\mathbf{R}^n)$ defined by

$$\langle w_\theta, \varphi \rangle = \frac{1}{\pi} \,\text{p.v.} \int_{-\infty}^{+\infty} \frac{\varphi(t\theta)}{t}\, dt.$$

Compute the Fourier transform of w_θ and prove that $\mathscr{H}_\theta$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$.
$\big[$*Hint:* Use that H maps $L^1(\mathbf{R})$ to $L^{1,\infty}(\mathbf{R})$, which follows from Theorem 5.3.3.$\big]$

5.2.2. Extend the definitions of W_Ω and T_Ω to $\Omega = d\mu$ a finite signed Borel measure on $\mathbf{S}^{n-1}$ with mean value zero. Compute the Fourier transform of $W_{d\mu}$ and find a necessary and sufficient condition on measures $d\mu$ so that $T_{d\mu}$ is L^2 bounded. Notice that the directional Hilbert transform $\mathscr{H}_\theta$ is a special case of such an operator $T_{d\mu}$.

5.2.3. Use the inequality $AB \le A\log A + e^B$ for $A \ge 1$ and $B > 0$ to prove that if Ω satisfies (5.2.24) then it must satisfy (5.2.16). Conclude that if $|\Omega|\log^+|\Omega|$ is in $L^1(\mathbf{S}^{n-1})$, then T_Ω is L^2 bounded.
$\big[$*Hint:* Use that $\int_{\mathbf{S}^{n-1}} |\xi \cdot \theta|^{-\alpha}\, d\theta$ converges when $\alpha < 1$. See Appendix D.3.$\big]$

5.2.4. Let Ω be a nonzero integrable function on $\mathbf{S}^{n-1}$ with mean value zero. Let f be integrable over $\mathbf{R}^n$ with nonzero integral. Prove that $T_\Omega(f)$ is not in $L^1(\mathbf{R}^n)$.
$\big[$*Hint:* Show that $\widehat{T_\Omega(f)}$ cannot be continuous at zero.$\big]$

5.2.5. Let $\theta \in \mathbf{S}^{n-1}$. Use an identity similar to (5.2.18) to show that the maximal operators

$$\sup_{a>0} \frac{1}{a}\int_0^a |f(x - r\theta)|\, dr, \qquad \sup_{a>0} \frac{1}{2a}\int_{-a}^{+a} |f(x - r\theta)|\, dr$$

are $L^p(\mathbf{R}^n)$ bounded for $1 < p < \infty$ with norm at most $3\,p\,(p-1)^{-1}$.

5.2.6. For $\Omega \in L^1(\mathbf{S}^{n-1})$ and f locally integrable on $\mathbf{R}^n$, define

$$M_\Omega(f)(x) = \sup_{R>0} \frac{1}{v_n R^n} \int_{|y| \le R} |\Omega(y/|y|)|\,|f(x-y)|\, dy.$$

Apply the method of rotations to prove that M_Ω maps $L^p(\mathbf{R}^n)$ to itself for $1 < p < \infty$.

5.2.7. Let $\Omega(x,\theta)$ be a function on $\mathbf{R}^n \times \mathbf{S}^{n-1}$ satisfying
(a) $\Omega(x,-\theta) = -\Omega(x,\theta)$ for all x and θ.
(b) $\sup_x |\Omega(x,\theta)|$ is in $L^1(\mathbf{S}^{n-1})$.

Use the method of rotations to prove that

$$T_\Omega(f)(x) = \text{p.v.}\int_{\mathbf{R}^n} \frac{\Omega(x, y/|y|)}{|y|^n} f(x-y)\,dy$$

is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

5.2.8. Let $\Omega \in L^1(\mathbf{S}^{n-1})$ have mean value zero. Prove that if T_Ω maps $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, then $p = q$.
$\big[$*Hint:* Use dilations.$\big]$

5.2.9. Prove that for all $1 < p < \infty$ there exists a constant $A_p > 0$ such that for every complex-valued $\mathscr{C}^2(\mathbf{R}^2)$ function f with compact support we have the bound

$$\big\|\partial_{x_1} f\big\|_{L^p} + \big\|\partial_{x_2} f\big\|_{L^p} \le A_p \big\|\partial_{x_1} f + i\partial_{x_2} f\big\|_{L^p}.$$

5.2.10. (a) Let $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ be the usual Laplacian on $\mathbf{R}^n$. Prove that for all $1 < p < \infty$ there exists a constant $A_p > 0$ such that for all $\mathscr{C}^2$ functions f with compact support we have the bound

$$\big\|\partial_{x_j}\partial_{x_k} f\big\|_{L^p} \le A_p \big\|\Delta f\big\|_{L^p}.$$

(b) Let $\Delta^m = \overbrace{\Delta \circ \cdots \circ \Delta}^{m \text{ times}}$. Show that for any $1 < p < \infty$ there exists a $C_p > 0$ such that for all f of class $\mathscr{C}^{2m}$ with compact support and all differential monomials ∂_x^α of order $|\alpha| = 2m$ we have

$$\big\|\partial_x^\alpha f\big\|_{L^p} \le C_p \big\|\Delta^m f\big\|_{L^p}.$$

5.2.11. Use the same idea as in Lemma 5.2.5 to show that if f is continuous on $[0,\infty)$, differentiable in $(0,\infty)$, and satisfies

$$\lim_{N\to\infty} \int_N^{Na} \frac{f(u)}{u}\,du = 0$$

for all $a > 0$, then

$$\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \frac{f(at) - f(t)}{t}\,dt = f(0)\log\frac{1}{a}\,.$$

5.2.12. Let Ω_o be an odd integrable function on $\mathbf{S}^{n-1}$ and Ω_e an even function on $\mathbf{S}^{n-1}$ that satisfies (5.2.24). Let f be a function supported in a ball B in $\mathbf{R}^n$. Prove that
(a) If $|f|\log^+|f|$ is integrable over a ball B, then $T_{\Omega_o}(f)$ and $T_{\Omega_o}^{(**)}(f)$ are integrable over B.
(b) If $|f|(\log^+|f|)^2$ is integrable over a ball B, then $T_{\Omega_e}(f)$ and $T_{\Omega_e}^{(**)}(f)$ are integrable over B.
$\big[$*Hint:* Use Exercise 1.3.7.$\big]$

5.2.13. ([324]) Let Ω be integrable on $\mathbf{S}^{n-1}$ with mean value zero. Use Jensen's inequality to show that for some $C > 0$ and every radial function $f \in L^2(\mathbf{R}^n)$ we have

$$\big\|T_\Omega(f)\big\|_{L^2} \le C\big\|f\big\|_{L^2}.$$

This inequality subsumes that T_Ω is well defined on radial $L^2(\mathbf{R}^n)$ functions.

5.3 The Calderón–Zygmund Decomposition and Singular Integrals

The behavior of singular integral operators on $L^1(\mathbf{R}^n)$ is a more subtle issue than that on L^p for $1 < p < \infty$. It turns out that singular integrals are not bounded from L^1 to L^1. See Example 5.1.3 and also Exercise 5.2.4. In this section we see that singular integrals map L^1 into the larger space $L^{1,\infty}$. This result strengthens their L^p boundedness.

5.3.1 The Calderón–Zygmund Decomposition

To make some advances in the theory of singular integrals, we need to introduce the Calderón–Zygmund decomposition. This is a powerful stopping-time construction that has many other interesting applications. We have already encountered an example of a stopping-time argument in Section 2.1.

Recall that a dyadic cube in $\mathbf{R}^n$ is the set

$$[2^k m_1, 2^k(m_1+1)) \times \cdots \times [2^k m_n, 2^k(m_n+1)),$$

where $k, m_1, \dots, m_n \in \mathbf{Z}$. Two dyadic cubes are either disjoint or related by inclusion.

Theorem 5.3.1. *Let $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$. Then there exist functions g and b on $\mathbf{R}^n$ such that*

(1) $f = g + b$.

(2) $\|g\|_{L^1} \le \|f\|_{L^1}$ *and* $\|g\|_{L^\infty} \le 2^n\alpha$.

(3) $b = \sum_j b_j$, *where each b_j is supported in a dyadic cube Q_j. Furthermore, the cubes Q_k and Q_j are disjoint when $j \neq k$.*

(4) $\displaystyle\int_{Q_j} b_j(x)\,dx = 0$.

(5) $\|b_j\|_{L^1} \le 2^{n+1}\alpha|Q_j|$.

(6) $\sum_j |Q_j| \le \alpha^{-1}\|f\|_{L^1}$.