Theorem 1.3.2. *Let (X,μ) and (Y,ν) be a pair of σ-finite measure spaces and let $0<p_0<p_1\le\infty$. Let T be a sublinear operator defined on $L^{p_0}(X)+L^{p_1}(X)=\{f_0+f_1:\ f_j\in L^{p_j}(X),\, j=0,1\}$ and taking values in the space of measurable functions on Y. Assume that there exist $A_0,A_1<\infty$ such that*

$$\big\|T(f)\big\|_{L^{p_0,\infty}(Y)}\le A_0\big\|f\big\|_{L^{p_0}(X)} \qquad \textit{for all } f\in L^{p_0}(X), \tag{1.3.5}$$

$$\big\|T(f)\big\|_{L^{p_1,\infty}(Y)}\le A_1\big\|f\big\|_{L^{p_1}(X)} \qquad \textit{for all } f\in L^{p_1}(X). \tag{1.3.6}$$

Then for all $p_0<p<p_1$ and for all f in $L^p(X)$ we have the estimate

$$\big\|T(f)\big\|_{L^p(Y)}\le A\big\|f\big\|_{L^p(X)}, \tag{1.3.7}$$

where

$$A=2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}. \tag{1.3.8}$$

Proof. Assume first that $p_1<\infty$. Fix f a function in $L^p(X)$ and $\alpha>0$. We split $f=f_0^\alpha+f_1^\alpha$, where f_0^α is in L^{p_0} and f_1^α is in L^{p_1}. The splitting is obtained by cutting $|f|$ at height $\delta\alpha$ for some $\delta>0$ to be determined later. Set

$$f_0^\alpha(x)\;=\begin{cases} f(x) & \text{for}\quad |f(x)|>\delta\alpha,\\ 0 & \text{for}\quad |f(x)|\le\delta\alpha,\end{cases}$$

$$f_1^\alpha(x)\;=\begin{cases} f(x) & \text{for}\quad |f(x)|\le\delta\alpha,\\ 0 & \text{for}\quad |f(x)|>\delta\alpha.\end{cases}$$

It can be checked easily that f_0^α (the unbounded part of f) is an L^{p_0} function and that f_1^α (the bounded part of f) is an L^{p_1} function. Indeed, since $p_0<p$, we have

$$\big\|f_0^\alpha\big\|_{L^{p_0}}^{p_0}=\int_{|f|>\delta\alpha}|f(x)|^p|f(x)|^{p_0-p}\,d\mu(x)\le(\delta\alpha)^{p_0-p}\big\|f\big\|_{L^p}^p$$

and similarly, since $p<p_1$,

$$\big\|f_1^\alpha\big\|_{L^{p_1}}^{p_1}\le(\delta\alpha)^{p_1-p}\big\|f\big\|_{L^p}^p.$$

In view of the subadditivity property of T contained in (1.3.3) we obtain that

$$|T(f)|\le|T(f_0^\alpha)|+|T(f_1^\alpha)|,$$

which implies

$$\{y\in Y:\,|T(f)(y)|>\alpha\}\subseteqq\{y\in Y:\,|T(f_0^\alpha)(y)|>\alpha/2\}\cup\{y\in Y:\,|T(f_1^\alpha)(y)|>\alpha/2\},$$

and therefore

$$d_{T(f)}(\alpha)\le d_{T(f_0^\alpha)}(\alpha/2)+d_{T(f_1^\alpha)}(\alpha/2). \tag{1.3.9}$$

Hypotheses (1.3.5) and (1.3.6) together with (1.3.9) now give

$$d_{T(f)}(\alpha) \le \frac{A_0^{p_0}}{(\alpha/2)^{p_0}} \int_{|f|>\delta\alpha} |f(x)|^{p_0}\, d\mu(x) + \frac{A_1^{p_1}}{(\alpha/2)^{p_1}} \int_{|f|\le\delta\alpha} |f(x)|^{p_1}\, d\mu(x).$$

In view of the last estimate and Proposition 1.1.4 (which can be used since Y is σ-finite), we obtain that

$$\begin{aligned}
\big\|T(f)\big\|_{L^p}^p &\le p(2A_0)^{p_0} \int_0^\infty \alpha^{p-1}\alpha^{-p_0} \int_{|f|>\delta\alpha} |f(x)|^{p_0}\, d\mu(x)\, d\alpha \\
&\qquad + p(2A_1)^{p_1} \int_0^\infty \alpha^{p-1}\alpha^{-p_1} \int_{|f|\le\delta\alpha} |f(x)|^{p_1}\, d\mu(x)\, d\alpha \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{\frac{1}{\delta}|f(x)|} \alpha^{p-1-p_0}\, d\alpha\, d\mu(x) \\
&\qquad + p(2A_1)^{p_1} \int_X |f(x)|^{p_1} \int_{\frac{1}{\delta}|f(x)|}^\infty \alpha^{p-1-p_1}\, d\alpha\, d\mu(x) \\
&= \frac{p(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} \int_X |f(x)|^{p_0} |f(x)|^{p-p_0}\, d\mu(x) \\
&\qquad + \frac{p(2A_1)^{p_1}}{p_1-p} \frac{1}{\delta^{p-p_1}} \int_X |f(x)|^{p_1} |f(x)|^{p-p_1}\, d\mu(x) \\
&= p\left(\frac{(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p} \delta^{p_1-p} \right) \big\|f\big\|_{L^p}^p,
\end{aligned}$$

and the convergence of the integrals in α is justified from $p_0 < p < p_1$, while the interchange of the integrals (Fubini's theorem) uses the hypothesis that (X,μ) is a σ-finite measure space. We pick $\delta > 0$ such that

$$(2A_0)^{p_0} \frac{1}{\delta^{p-p_0}} = (2A_1)^{p_1} \delta^{p_1-p},$$

and observe that the last displayed constant is equal to the pth power of the constant in (1.3.8). We have therefore proved the theorem when $p_1 < \infty$.

We now consider the case $p_1 = \infty$. Write $f = f_0^\alpha + f_1^\alpha$, where

$$\begin{aligned}
f_0^\alpha(x) &= \begin{cases} f(x) & \text{for} \quad |f(x)| > \gamma\alpha, \\ 0 & \text{for} \quad |f(x)| \le \gamma\alpha, \end{cases} \\
f_1^\alpha(x) &= \begin{cases} f(x) & \text{for} \quad |f(x)| \le \gamma\alpha, \\ 0 & \text{for} \quad |f(x)| > \gamma\alpha. \end{cases}
\end{aligned}$$

We have

$$\big\|T(f_1^\alpha)\big\|_{L^\infty} \le A_1 \big\|f_1^\alpha\big\|_{L^\infty} \le A_1\gamma\alpha = \alpha/2,$$