$$= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbf{R}^n} \varphi(\xi) \int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{\varepsilon\le r\le N} \left(e^{-2\pi r|\xi| i\theta\cdot\xi'} - \cos(2\pi r|\xi|)\right)\frac{dr}{r}\, d\theta\, d\xi$$

$$= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbf{R}^n} \varphi(\xi) \int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{2\pi|\xi|\varepsilon}^{2\pi|\xi|N} \frac{e^{-ir\theta\cdot\xi'} - \cos(r)}{r}\, dr\, d\theta\, d\xi$$

$$= \int_{\mathbf{R}^n} \varphi(\xi) \int_{\mathbf{S}^{n-1}} \Omega(\theta) \left(\log\frac{1}{|\xi'\cdot\theta|} - \frac{i\pi}{2}\operatorname{sgn}(\xi\cdot\theta)\right) d\theta\, d\xi,$$

where we used the Lebesgue dominated convergence theorem to pass the limit inside, Lemma 5.2.5, and Remark 5.2.4. We were able to subtract $\cos(2\pi r|\xi|)$ from the r integral in the previous calculation, since Ω has mean value zero over the sphere. Also, the use of the dominated convergence theorem is justified from the fact that the function

$$(\theta,\xi) \mapsto |\Omega(\theta)||\varphi(\xi)|\left(\log\frac{1}{|\xi'\cdot\theta|} + 4\right)$$

lies in $L^1(\mathbf{S}^{n-1}\times\mathbf{R}^n)$. Moreover, all the interchanges of integrals are well justified by Fubini's theorem. □

Corollary 5.2.6. *Let $\Omega \in L^1(\mathbf{S}^{n-1})$ have mean value zero. Then for almost all ξ' in $\mathbf{S}^{n-1}$ the integral*

$$\int_{\mathbf{S}^{n-1}} \Omega(\theta)\log\frac{1}{|\xi'\cdot\theta|}\, d\theta \tag{5.2.15}$$

converges absolutely. Moreover, the associated operator T_Ω maps $L^2(\mathbf{R}^n)$ to itself if and only if

$$\operatorname*{ess.sup}_{\xi'\in\mathbf{S}^{n-1}} \left| \int_{\mathbf{S}^{n-1}} \Omega(\theta)\log\frac{1}{|\xi'\cdot\theta|}\, d\theta \right| < \infty. \tag{5.2.16}$$

Proof. To obtain the absolute convergence of the integral in (5.2.15) we integrate over $\xi' \in \mathbf{S}^{n-1}$ and we apply Fubini's theorem. The assertion concerning the boundedness of T_Ω on L^2 is an immediate consequence of Proposition 5.2.3 and Theorem 2.5.10. □

There exist functions Ω in $L^1(\mathbf{S}^{n-1})$ with mean value zero such that the expressions in (5.2.16) are equal to infinity; consequently, not all such Ω give rise to bounded operators on $L^2(\mathbf{R}^n)$. Observe, however, that for Ω odd i.e., $\Omega(-\theta) = -\Omega(\theta)$ for all $\theta \in \mathbf{S}^{n-1}$, (5.2.16) trivially holds, since $\log\frac{1}{|\xi\cdot\theta|}$ is even and its product against an odd function must have integral zero over $\mathbf{S}^{n-1}$. We conclude that singular integrals T_Ω with odd Ω are always L^2 bounded.