for $\varphi \in \mathscr{S}(\mathbf{R}^n)$. Using the fact that Ω has mean value zero, we can easily see that W_Ω is a well defined tempered distribution on $\mathbf{R}^n$. Indeed, since K_Ω has integral zero over all annuli centered at the origin, we have

$$\begin{aligned}
\big|\langle W_\Omega, \varphi\rangle\big| &= \left|\lim_{\varepsilon\to 0}\int_{\varepsilon\le|x|\le 1}\frac{\Omega(x/|x|)}{|x|^n}(\varphi(x)-\varphi(0))\,dx+\int_{|x|\ge 1}\frac{\Omega(x/|x|)}{|x|^n}\varphi(x)\,dx\right| \\
&\le \big\|\nabla\varphi\big\|_{L^\infty}\int_{|x|\le 1}\frac{|\Omega(x/|x|)|}{|x|^{n-1}}\,dx+\sup_{y\in\mathbf{R}^n}|y|\,|\varphi(y)|\int_{|x|\ge 1}\frac{|\Omega(x/|x|)|}{|x|^{n+1}}\,dx \\
&\le C_1\big\|\nabla\varphi\big\|_{L^\infty}\big\|\Omega\big\|_{L^1}+C_2\sum_{|\alpha|\le 1}\big\|\varphi(x)x^\alpha\big\|_{L^\infty}\big\|\Omega\big\|_{L^1},
\end{aligned}$$

for suitable C_1 and C_2, where we used (2.2.2) in the last estimate. Note that the distribution W_Ω coincides with the function K_Ω on $\mathbf{R}^n\setminus\{0\}$.

The kernels of the Hilbert transform and the Riesz transforms are examples of ~~these general operators~~ K_Ω. For instance, the function $\Omega(\theta)=\frac{\theta}{\pi|\theta|}=\frac{1}{\pi}\operatorname{sgn}\theta$ defined on the unit sphere $\mathbf{S}^0=\{-1,1\}\subsetneqq\mathbf{R}$ gives rise to the Hilbert transform, while the function

$$\Omega(\theta)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{\theta_j}{|\theta|}$$

defined on $\mathbf{S}^{n-1}\subsetneqq\mathbf{R}^n$ gives rise to the jth Riesz transform.

Definition 5.2.1. Let Ω be integrable on the sphere $\mathbf{S}^{n-1}$ with mean value zero. For $0<\varepsilon<N$ and $f\in\bigcup_{1\le p<\infty}L^p(\mathbf{R}^n)$ we define the *truncated singular integral*

$$T_\Omega^{(\varepsilon,N)}(f)(x)=\int_{\varepsilon\le|y|\le N}f(x-y)\,\frac{\Omega(y/|y|)}{|y|^n}\,dy. \tag{5.2.3}$$

Note that for $f\in L^p(\mathbf{R}^n)$ we have

$$\big\|T_\Omega^{(\varepsilon,N)}(f)\big\|_{L^p}\le\big\|\Omega\big\|_{L^1}\log(N/\varepsilon)\,\big\|f\big\|_{L^p(\mathbf{R}^n)},$$

which implies that (5.2.3) is finite a.e. and therefore well defined. We denote by T_Ω the singular integral operator whose kernel is the distribution W_Ω, that is,

$$T_\Omega(f)(x)=(f*W_\Omega)(x)=\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}T_\Omega^{(\varepsilon,N)}(f)(x),$$

defined for $f\in\mathscr{S}(\mathbf{R}^n)$. The associated *maximal singular integral* is defined by

$$T_\Omega^{(**)}(f)=\sup_{0<N<\infty}\sup_{0<\varepsilon<N}\big|T_\Omega^{(\varepsilon,N)}(f)\big|. \tag{5.2.4}$$