Proposition 5.1.17. *For φ in $\mathscr{S}(\mathbf{R}^n)$ and $1 \le j,k \le n$ we have*

$$\partial_j \partial_k \varphi(x) = -R_j R_k \Delta \varphi(x) \tag{5.1.47}$$

for all $x \in \mathbf{R}^n$.

Proof. We verify the claimed identity by taking Fourier transforms. We have

$$\begin{aligned}\big(\partial_j \partial_k \varphi\big)^{\widehat{}}(\xi) &= (2\pi i \xi_j)(2\pi i \xi_k)\widehat{\varphi}(\xi) \\ &= -\Big(-\frac{i\xi_j}{|\xi|}\Big)\Big(-\frac{i\xi_k}{|\xi|}\Big)(-4\pi^2|\xi|^2)\widehat{\varphi}(\xi) \\ &= -\big(R_j R_k \Delta \varphi\big)^{\widehat{}}(\xi)\end{aligned}$$

and taking the inverse Fourier transform, identity (5.1.47) follows. □

Next we discuss a use of the Riesz transforms to partial differential equations.

Example 5.1.18. Suppose that f is a given function in $L^2(\mathbf{R}^n)$ and that u is a tempered distribution on $\mathbf{R}^n$ that solves *Laplace's equation*

$$\Delta u = f\,. \tag{5.1.48}$$

We express all second-order derivatives of u in terms of the Riesz transforms of f.

To solve equation (5.1.48) we first show that the tempered distribution

$$\big(\partial_j \partial_k u + R_j R_k(f)\big)^{\widehat{}}$$

is supported at $\{0\}$. In view of Proposition 2.4.1, this implies that

$$\partial_j \partial_k u = -R_j R_k(f) + P$$

where P is a polynomial of n variables (that depends on j and k) and provides a way to express the mixed partials of u in terms of the Riesz transforms of f.

To verify that $\big(\partial_j \partial_k u + R_j R_k(f)\big)^{\widehat{}}$ is supported at $\{0\}$, we fix a Schwartz function ψ whose support does not contain the origin. Then ψ vanishes in a neighborhood of zero and we can pick a $\mathscr{C}^\infty$ function η which vanishes in a smaller neighborhood of zero and is equal to 1 on the support of ψ. We define

$$\zeta(\xi) = -\eta(\xi)\Big(-\frac{i\xi_j}{|\xi|}\Big)\Big(-\frac{i\xi_k}{|\xi|}\Big)$$

and we notice that ζ is a bounded $\mathscr{C}^\infty$ function and so are all of its derivatives; also

$$\eta(\xi)(2\pi i \xi_j)(2\pi i \xi_k) = \zeta(\xi)(-4\pi^2|\xi|^2)\,.$$

Taking the Fourier transform of both sides of (5.1.48) we obtain

$$(-4\pi^2|\xi|^2)\,\widehat{u}(\xi) = \widehat{f}(\xi)$$