We now turn to the proof of (5.1.31). It suffices to prove (5.1.31) for Schwartz functions since, given $f \in L^p$ there is a sequence $\phi_j \in \mathscr{S}$ such that $\|f - \phi_j\|_{L^p} \to 0$ as $j \to \infty$ and P_ε, Q_ε lie in $L^{p'}$. Taking Fourier transforms, we see that (5.1.31) is a consequence of the identity

$$\big((-i\,\mathrm{sgn}\,\xi)e^{-2\pi|\xi|}\big)^\vee(x) = \frac{1}{\pi}\frac{x}{x^2+1}\,. \tag{5.1.36}$$

To prove (5.1.36) we write

$$\begin{aligned}
\big((-i\,\mathrm{sgn}\,\xi)e^{-2\pi|\xi|}\big)^\vee(x) &= \int_{-\infty}^{+\infty} e^{-2\pi|\xi|}(-i\,\mathrm{sgn}\,\xi)e^{2\pi i x\xi}\,d\xi \\
&= 2\int_0^\infty e^{-2\pi\xi}\sin(2\pi x\xi)\,d\xi \\
&= \frac{1}{\pi}\int_0^\infty e^{-\xi}\sin(x\xi)\,d\xi \qquad\qquad (5.1.37) \\
&= \frac{1}{\pi}\int_0^\infty (e^{-\xi})''\sin(x\xi)\,d\xi \\
&= -\frac{x}{\pi}\int_0^\infty (e^{-\xi})'\cos(x\xi)\,d\xi \\
&= -\frac{x}{\pi}\bigg[-1 + x\int_0^\infty e^{-\xi}\sin(x\xi)\,d\xi\bigg] \qquad (5.1.38)
\end{aligned}$$

and we equate (5.1.38) and (5.1.37).

The statement in the theorem about the almost everywhere convergence of $H^{(\varepsilon)}(f)$ to $H(f)$ is a consequence of (5.1.30), of the fact that the alleged convergence holds for Schwartz functions, and of Theorem 2.1.14. Finally, the L^p convergence follows from the almost everywhere convergence and the Lebesgue dominated convergence theorem in view of the validity of (5.1.35). □

5.1.4 The Riesz Transforms

In this section we fix $n \geq 2$ and we study an n-dimensional analogue of the Hilbert transform. It turns out that there exist n operators in $\mathbf{R}^n$, called the Riesz transforms, with properties analogous to those of the Hilbert transform on $\mathbf{R}$.

To define the Riesz transforms, we first introduce tempered distributions W_j on $\mathbf{R}^n$, for $1 \leq j \leq n$, as follows. For $\varphi \in \mathscr{S}(\mathbf{R}^n)$, let

$$\big\langle W_j, \varphi\big\rangle = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{y_j}{|y|^{n+1}}\varphi(y)\,dy.$$

One should check that indeed $W_j \in \mathscr{S}'(\mathbf{R}^n)$. Observe that the normalization of W_j is similar to that of the Poisson kernel.