This completes the induction. We have proved that H maps L^p to L^p when $p = 2^k$, $k = 1,2,\ldots$. Interpolation now gives that H maps L^p to L^p for all $p \ge 2$. Since $H^* = -H$, duality gives that H is also bounded on L^p for $1 < p \le 2$.

The previous proof of the boundedness of the Hilbert transform provides us with some useful information about the norm of this operator on $L^p(\mathbf{R})$. Let us begin with the identity

$$\cot\frac{x}{2} = \cot x + \sqrt{1+\cot^2 x},$$

valid for $0 < x \le \frac{\pi}{2}$. If $c_p \le \cot\frac{\pi}{2p}$, then (5.1.27) gives that

$$c_{2p} \le c_p + \sqrt{c_p^2+1} \le \cot\frac{\pi}{2p} + \sqrt{1+\cot^2\frac{\pi}{2p}} = \cot\frac{\pi}{2\cdot 2p},$$

and since $1 = \cot\frac{\pi}{4} = \cot\frac{\pi}{2\cdot 2}$, we obtain by induction that the numbers $\cot\frac{\pi}{2p}$ are indeed bounds for the norm of H on L^p when $p = 2^k$, $k = 1,2,\ldots$. Duality now gives that the numbers $\cot\frac{\pi}{2p'} = \tan\frac{\pi}{2p}$ are bounds for the norm of H on L^p when $p = \frac{2^k}{2^k-1}$, $k = 1,2,\ldots$. These bounds allow us to derive good estimates for the norm $\|H\|_{L^p\to L^p}$ as $p \to 1$ and $p \to \infty$. Indeed, since $\cot\frac{\pi}{2p} \le p$ when $p \ge 2$, the Riesz–Thorin interpolation theorem gives that $\|H\|_{L^p\to L^p} \le 2p$ for $2 \le p < \infty$ and by duality $\|H\|_{L^p\to L^p} \le \frac{2p}{p-1}$ for $1 < p \le 2$. This completes the proof which is worth comparing with that of Theorem 4.1.7. □

Remark 5.1.8. The numbers $\cot\frac{\pi}{2p}$ for $2 \le p < \infty$ and $\tan\frac{\pi}{2p}$ for $1 < p \le 2$ are indeed equal to the norms of the Hilbert transform H on $L^p(\mathbf{R})$. This requires a more delicate argument; see Exercise 5.1.12.

Remark 5.1.9. We may wonder what happens when $p = 1$ or $p = \infty$. The Hilbert transform of $\chi_{[a,b]}$ computed in Example 5.1.3 is easily seen to be unbounded and not integrable, since it behaves like $1/|x|$ as $x \to \infty$. This behavior near infinity suggests that the Hilbert transform may map L^1 to $L^{1,\infty}$. This is indeed the case, but this will not be shown until Section 5.3.

We now introduce the maximal Hilbert transform.

Definition 5.1.10. The *maximal Hilbert transform* is the operator

$$H^{(*)}(f)(x) = \sup_{\varepsilon>0}\left|H^{(\varepsilon)}(f)(x)\right| \tag{5.1.28}$$

defined for all f in L^p, $1 \le p < \infty$. For such f, $H^{(\varepsilon)}(f)$ is well defined as a convergent integral by Hölder's inequality. Hence $H^{(*)}(f)$ makes sense for $f \in L^p(\mathbf{R})$, although for some values of x, $H^{(*)}(f)(x)$ may be infinite.

Example 5.1.11. Using the result of Example 5.1.4, we obtain that

$$H^{(*)}(\chi_{[a,b]})(x) = \frac{1}{\pi}\left|\log\frac{|x-a|}{|x-b|}\right|. \tag{5.1.29}$$