be the symbol of the Hilbert transform. We have

$$
\begin{aligned}
&\widehat{f}^2(\xi) + 2[H(fH(f))]^\frown(\xi) \\
&\qquad = (\widehat{f} * \widehat{f})(\xi) + 2m(\xi)(\widehat{f} * \widehat{H(f)})(\xi) \\
&\qquad = \int_{\mathbf{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)\,d\eta + 2m(\xi)\int_{\mathbf{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)m(\eta)\,d\eta \qquad (5.1.25) \\
&\qquad = \int_{\mathbf{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)\,d\eta + 2m(\xi)\int_{\mathbf{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)m(\xi-\eta)\,d\eta. \qquad (5.1.26)
\end{aligned}
$$

Averaging (5.1.25) and (5.1.26) we obtain

$$
\widehat{f}^2(\xi) + 2[H(fH(f))]^\frown(\xi) = \int_{\mathbf{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)\big[1 + m(\xi)\big(m(\eta) + m(\xi-\eta)\big)\big]\,d\eta\,.
$$

But the last displayed expression is equal to

$$
\int_{\mathbf{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)m(\eta)m(\xi-\eta)\,d\eta = (\widehat{H(f)} * \widehat{H(f)})(\xi)
$$

in view of the identity

$$
m(\eta)m(\xi-\eta) = 1 + m(\xi)m(\eta) + m(\xi)m(\xi-\eta),
$$

which is valid for all $(\xi,\eta) \in \mathbf{R}^2 \setminus \{(0,0)\}$ for the function $m(\xi) = -i\operatorname{sgn}\xi$.

Having established (5.1.23), we can easily obtain L^p bounds for H when $p = 2^k$ is a power of 2. We already know that H is bounded on L^p with norm one when $p = 2^k$ and $k = 1$. Suppose that H is bounded on L^p with bound c_p for $p = 2^k$ for some $k \in \mathbf{Z}^+$. Then for a nonzero real-valued function f in $\mathscr{C}_0^\infty$ we have

$$
\begin{aligned}
\big\|H(f)\big\|_{L^{2p}}^2 = \big\|H(f)^2\big\|_{L^p} &\le \big\|f^2\big\|_{L^p} + \big\|2H(fH(f))\big\|_{L^p} \\
&\le \big\|f\big\|_{L^{2p}}^2 + 2c_p\big\|fH(f)\big\|_{L^p} \\
&\le \big\|f\big\|_{L^{2p}}^2 + 2c_p\big\|f\big\|_{L^{2p}}\big\|H(f)\big\|_{L^{2p}}.
\end{aligned}
$$

Dividing by $\|f\|_{L^{2p}} \neq 0$ and using that $\|H(f)\|_{L^{2p}} < \infty$, we obtain that

$$
\left(\frac{\|H(f)\|_{L^{2p}}}{\|f\|_{L^{2p}}}\right)^2 - 2c_p\frac{\|H(f)\|_{L^{2p}}}{\|f\|_{L^{2p}}} - 1 \le 0.
$$

If follows that

$$
\frac{\|H(f)\|_{L^{2p}}}{\|f\|_{L^{2p}}} \le c_p + \sqrt{c_p^2 + 1}\,,
$$

and from this we conclude that H is bounded on L^{2p} with bound

$$
c_{2p} \le c_p + \sqrt{c_p^2 + 1}\,. \qquad (5.1.27)
$$