(b) Use part (a) to show that $\{Q_k\}_k$ is an approximate identity on $\mathbf{R}$ as $k \to \infty$.
(c) Given a continuous function f on $\mathbf{R}$ that vanishes outside the interval $[0,1]$, show that $f * Q_k$ converges to f uniformly on $[0,1]$ as $k \to \infty$.
(d) (*Weierstrass*) Prove that every continuous function on $[0,1]$ can be approximated uniformly by polynomials.
$\big[$*Hint:* Part (a): Estimate the integral $\int_{|t|\le k^{-1/2}} Q_k(t)\,dt$ from below using the inequality $(1-t^2)^k \ge 1-kt^2$ for $|t| \le 1$. Part (d): Consider the function $g(t) = f(t) - f(0) - t(f(1)-f(0))$.$\big]$

1.2.12. Show that the *Laplace transform* $L(f)(x) = \int_0^\infty f(t)e^{-xt}\,dt$ maps $L^2(0,\infty)$ to itself with norm at most $\sqrt{\pi}$.
$\big[$*Hint:* Consider convolution with the kernel $\sqrt{t}\,e^{-t}$ on the group $L^2((0,\infty), \frac{dt}{t})$.$\big]$

1.2.13. ([62]) Let $F \ge 0$, $G \ge 0$ be measurable functions on the sphere $\mathbf{S}^{n-1}$ and let $K \ge 0$ be a measurable function on $[-1,1]$. Prove that

$$\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}} F(\theta)G(\varphi)K(\theta\cdot\varphi)\,d\varphi\,d\theta \le C\,\|F\|_{L^p(\mathbf{S}^{n-1})}\|G\|_{L^{p'}(\mathbf{S}^{n-1})},$$

where $1 \le p \le \infty$, $\theta\cdot\varphi = \sum_{j=1}^n \theta_j\varphi_j$ and $C = \int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\varphi$, which is independent of θ. Moreover, show that C is the best possible constant in the preceding inequality. Using duality, compute the norm of the linear operator

$$F(\theta) \mapsto \int_{\mathbf{S}^{n-1}} F(\varphi)K(\theta\cdot\varphi)\,d\varphi$$

from $L^p(\mathbf{S}^{n-1})$ to itself.
$\big[$*Hint:* Observe that $\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}} F(\theta)G(\varphi)K(\theta\cdot\varphi)\,d\varphi\,d\theta$ is bounded by the quantity

$$\left\{\int_{\mathbf{S}^{n-1}}\left[\int_{\mathbf{S}^{n-1}} F(\theta)K(\theta\cdot\varphi)\,d\theta\right]^p d\varphi\right\}^{\frac{1}{p}} \|G\|_{L^{p'}(\mathbf{S}^{n-1})}.$$

Apply Hölder's inequality to the functions F and 1 with respect to the measure $K(\theta\cdot\varphi)\,d\theta$ to deduce that $\int_{\mathbf{S}^{n-1}} F(\theta)K(\theta\cdot\varphi)\,d\theta$ is controlled by

$$\left(\int_{\mathbf{S}^{n-1}} F(\theta)^p K(\theta\cdot\varphi)\,d\theta\right)^{1/p}\left(\int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\theta\right)^{1/p'}.$$

Use Fubini's theorem to bound the latter by

$$\|F\|_{L^p(\mathbf{S}^{n-1})}\|G\|_{L^{p'}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\varphi.$$

Note that equality is attained if and only if both F and G are constants.$\big]$