Note that ψ is integrable over the line and has integral zero. Furthermore, the integrable function

$$\Psi(t) = \begin{cases} \frac{1}{t^2+1} & \text{when } |t| \geq 1, \\ 1 & \text{when } |t| < 1, \end{cases} \tag{5.1.21}$$

is a radially decreasing majorant of ψ, i.e., it is even, decreasing on $[0,\infty)$, and satisfies $|\psi| \leq \Psi$. It follows from Theorem 1.2.21 (with $a = 0$) that $f * \psi_\varepsilon \to 0$ in L^p. Also Corollary 2.1.19 (with $a = 0$) implies that $f * \psi_\varepsilon \to 0$ almost everywhere as $\varepsilon \to 0$.

Assertion (5.1.19) is a consequence of (5.1.18), the discussion preceding Theorem 5.1.5, and the observation that $H^{(\varepsilon)}(\varphi)$ converges to $H(\varphi)$ pointwise everywhere as $\varepsilon \to 0$. □

Remark 5.1.6. We will show later that for $f \in L^p(\mathbf{R})$, $1 \leq p < \infty$, the expressions $H^{(\varepsilon)}(f)$ converge a.e. (and also in L^p when $p > 1$) to a function $\widetilde{H}(f)$. This will be a consequence of Theorem 5.1.12 (or Corollary 5.3.6 when $p = 1$), combined with Theorem 2.1.14 and the observation that for Schwartz functions φ, $H^{(\varepsilon)}(\varphi)$ converge to $H(\varphi)$ as $\varepsilon \to 0$. The linear operator $\widetilde{H}$ defined in this way extends the Hilbert transform H initially defined on Schwartz functions and will still be denoted by H. Thus for $f \in L^p(\mathbf{R})$, $1 \leq p < \infty$, one has

$$\lim_{\varepsilon \to 0} f * Q_\varepsilon = H(f) \quad \text{a.e.}$$

This convergence is also valid in L^p in view of the preceding observations and Theorem 5.1.5.

5.1.3 *L^p Boundedness of the Hilbert Transform*

As a consequence of the result in Exercise 5.1.4 and of the fact that

$$x \leq \tfrac{1}{2}(e^x - e^{-x}), \qquad x \geq 0,$$

we obtain that

$$|\{x : \ |H(\chi_E)(x)| > \alpha\}| \leq \frac{2}{\pi} \frac{|E|}{\alpha}, \qquad \alpha > 0, \tag{5.1.22}$$

for all subsets E of the real line of finite measure. Theorem 1.3.2 with $p_0 = 1$ and $p_1 = 2$ combined with Remark 1.3.3 imply that H is bounded on L^p for $1 < p < 2$. Duality gives that $H^* = -H$ is bounded on L^p for $2 < p < \infty$ and hence so is H.

We give another proof of the boundedness of the Hilbert transform H on $L^p(\mathbf{R})$, which has the advantage that it gives the best possible constant in the resulting norm inequality when p is a power of 2.