This implies that

$$\widehat{W_0}(\xi) = -i\,\mathrm{sgn}\,\xi\,. \tag{5.1.12}$$

In particular, identity (5.1.12) says that $\widehat{W_0}$ is a (bounded) function.

We now use identity (5.1.12) to write

$$H(f)(x) = \big(\widehat{f}(\xi)(-i\,\mathrm{sgn}\,\xi)\big)^{\vee}(x)\,. \tag{5.1.13}$$

This formula can be used to give an alternative definition of the Hilbert transform. An immediate consequence of (5.1.13) is that

$$\big\|H(f)\big\|_{L^2} = \big\|f\big\|_{L^2}\,, \tag{5.1.14}$$

that is, H is an isometry on $L^2(\mathbf{R})$. Moreover, H satisfies

$$H^2 = HH = -I\,, \tag{5.1.15}$$

where I is the identity operator. Equation (5.1.15) is a simple consequence of the fact that $(-i\,\mathrm{sgn}\,\xi)^2 = -1$. The adjoint operator H^* of H is uniquely defined via the identity

$$\big\langle f\,|\,H(g)\big\rangle = \int_{\mathbf{R}} f\,\overline{H(g)}\,dx = \int_{\mathbf{R}} H^*(f)\,\overline{g}\,dx = \big\langle H^*(f)\,|\,g\big\rangle\,,$$

and we can easily obtain that H^* has multiplier $\overline{-i\,\mathrm{sgn}\,\xi} = i\,\mathrm{sgn}\,\xi$. We conclude that $H^* = -H$. Likewise, we obtain $H^t = -H$.

5.1.2 Connections with Analytic Functions

We now investigate connections of the Hilbert transform with the Poisson kernel. Recall the definition of the Poisson kernel P_y given in Example 1.2.16. Then for a real-valued function f in $L^p(\mathbf{R})$, $1 \le p < \infty$, we have

$$(P_y * f)(x) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{(x-t)^2+y^2}\,dt\,, \tag{5.1.16}$$

and the integral in (5.1.16) converges absolutely by Hölder's inequality, since the function $t \mapsto ((x-t)^2+y^2)^{-1}$ is in $L^{p'}(\mathbf{R})$ whenever $y > 0$.

Let $\mathrm{Re}\,z$ and $\mathrm{Im}\,z$ denote the real and imaginary parts of a complex number z. Observe that

$$(P_y * f)(x) = \mathrm{Re}\left(\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{x-t+iy}\,dt\right) = \mathrm{Re}\left(\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{z-t}\,dt\right),$$