which yields (5.1.6). Observe that the cancellation of ε in (5.1.7) reflects the fact that $1/x$ has integral zero on symmetric intervals $\varepsilon < |x| < c$. Note that $H(\chi_{[a,b]})(x)$ blows up logarithmically in x near the points a and b and decays like $|x|^{-1}$ as $x \to \infty$. See Figure 5.1.

Example 5.1.4. Let $\log^+ x = \log x$ when $x \geq 1$ and zero otherwise. Observe that the calculation in the previous example actually gives

$$H^{(\varepsilon)}(\chi_{[a,b]})(x) = \begin{cases} \dfrac{1}{\pi}\log^+ \dfrac{|x-a|}{\max(\varepsilon, |x-b|)} & \text{when } x > b, \\[2ex] -\dfrac{1}{\pi}\log^+ \dfrac{|x-b|}{\max(\varepsilon, |x-a|)} & \text{when } x < a, \\[2ex] \dfrac{1}{\pi}\log^+ \dfrac{|x-a|}{\varepsilon} - \dfrac{1}{\pi}\log^+ \dfrac{|x-b|}{\varepsilon} & \text{when } a < x < b. \end{cases}$$

We now give an alternative characterization of the Hilbert transform using the Fourier transform. To achieve this we need to compute the Fourier transform of the distribution W_0 defined in (5.1.1). Fix a Schwartz function φ on $\mathbf{R}$. Then

$$\begin{aligned} \big\langle \widehat{W_0}, \varphi \big\rangle &= \big\langle W_0, \widehat{\varphi} \big\rangle && (5.1.8) \\ &= \frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{|\xi|\geq\varepsilon} \widehat{\varphi}(\xi)\,\frac{d\xi}{\xi} \\ &= \frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{\frac{1}{\varepsilon}\geq|\xi|\geq\varepsilon} \int_{\mathbf{R}} \varphi(x) e^{-2\pi i x\xi}\,dx\,\frac{d\xi}{\xi} \\ &= \lim_{\varepsilon\to 0} \int_{\mathbf{R}} \varphi(x) \left[\frac{1}{\pi} \int_{\frac{1}{\varepsilon}\geq|\xi|\geq\varepsilon} e^{-2\pi i x\xi}\,\frac{d\xi}{\xi} \right] dx \\ &= \lim_{\varepsilon\to 0} \int_{\mathbf{R}} \varphi(x) \left[\frac{-i}{\pi} \int_{\frac{1}{\varepsilon}\geq|\xi|\geq\varepsilon} \sin(2\pi x\xi)\,\frac{d\xi}{\xi} \right] dx \\ &= \lim_{\varepsilon\to 0} \int_{\mathbf{R}} \varphi(x) \left[\left(\frac{-i}{\pi} \operatorname{sgn} x \right) \int_{\frac{2\pi}{\varepsilon}\geq|\xi|\geq 2\pi\varepsilon} \sin(|x|\xi)\,\frac{d\xi}{\xi} \right] dx\,. && (5.1.9) \end{aligned}$$

Here we used the signum function

$$\operatorname{sgn} x = \begin{cases} +1 & \text{when } x > 0, \\ 0 & \text{when } x = 0, \\ -1 & \text{when } x < 0. \end{cases} \tag{5.1.10}$$

Using the results (a) and (b) in Exercise 5.1.1 we obtain that the integrals inside the square brackets in (5.1.9) are uniformly bounded by 8 and converge to $2\frac{\pi}{2} = \pi$ as $\varepsilon \to 0$, whenever $x \neq 0$. These observations make possible the use of the Lebesgue dominated convergence theorem that allows the passage of the limit inside the integral in (5.1.9). We obtain that

$$\big\langle \widehat{W_0}, \varphi \big\rangle = \int_{\mathbf{R}} \varphi(x)(-i\operatorname{sgn}(x))\,dx\,. \tag{5.1.11}$$