We have

$$
\begin{aligned}
&\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}\sup_{0<N\le L}\bigg|\sum_{|m_1|\le N}\sum_{|m_2|\le N}\widehat{f}(m_1,m_2)e^{2\pi i m_1x_1}e^{2\pi i m_2x_2}\bigg|^p\,dx_2\,dx_1\\
&\quad\le 2^{p-1}\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}\sup_{0<N\le L}\bigg|\sum_{|m_2|\le N}\bigg[\sum_{|m_1|\le|m_2|}\widehat{f}(m_1,m_2)e^{2\pi i m_1x_1}\bigg]e^{2\pi i m_2x_2}\bigg|^p\\
&\qquad\qquad+\sup_{0<N\le L}\bigg|\sum_{|m_1|\le N}\bigg[\sum_{|m_2|<|m_1|}\widehat{f}(m_1,m_2)e^{2\pi i m_2x_2}\bigg]e^{2\pi i m_1x_1}\bigg|^p\,dx_1\,dx_2\\
&\quad=2^{p-1}\bigg[\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}\sup_{0<N\le L}\big|(D_N*f^L_{x_1})(x_2)\big|^p\,dx_2\,dx_1\\
&\qquad\qquad+\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}\sup_{0<N\le L}\big|(D_N*f^{x_2}_L)(x_1)\big|^p\,dx_1\,dx_2\bigg]\\
&\quad\le 2^{p-1}\bigg[\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}\sup_{N\in\mathbf{Z}^+}\big|(D_N*f^L_{x_1})(x_2)\big|^p\,dx_2dx_1\\
&\qquad\qquad+\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}\sup_{N\in\mathbf{Z}^+}\big|(D_N*f^{x_2}_L)(x_1)\big|^p\,dx_1\,dx_2\bigg]\\
&\quad\le 2^{p-1}C_p^p\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}|f^L_{x_1}(x_2)|^p\,dx_2\,dx_1+2^{p-1}C_p^p\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}|f^{x_2}_L(x_1)|^p\,dx_1\,dx_2\\
&\quad\le 2^pC_p^pB_p^p\big\|f\big\|^p_{L^p(\mathbf{T}^2)},
\end{aligned}
$$

where we used Theorem 4.3.14 in the penultimate inequality and estimate (4.3.45) in the last inequality. Since the last estimate we obtained is independent of $L\in\mathbf{Z}^+$, letting $L\to\infty$ and applying Fatou's lemma, we obtain the conclusion (4.3.44) for $n=2$. When $n\ge 3$ the idea of the proof is similar, but the notation a bit more cumbersome. $\square$

Exercises

4.3.1. Let $\alpha\ge 0$ and $\varphi\in\mathscr{C}_0^\infty$, equal to 0 on $B(0,\frac{1}{8})$, and 1 on $B(0,\frac{1}{4})^c$. Prove that $\varphi(\xi)(1-|\xi|^2)_+^\alpha$ is in $\mathscr{M}_p(\mathbf{R}^n)$ if and only if $\varphi(\xi)(1-|\xi|)_+^\alpha$ is in $\mathscr{M}_p(\mathbf{R}^n)$.
[*Hint:* Use that smooth functions with compact support lie in $\mathscr{M}_p(\mathbf{R}^n)$.]

4.3.2. The purpose of this exercise is to introduce distributions on the torus. The set of test functions on the torus is $\mathscr{C}^\infty(\mathbf{T}^n)$ equipped with the following topology. Given f_j, f in $\mathscr{C}^\infty(\mathbf{T}^n)$, we say that $f_j\to f$ in $\mathscr{C}^\infty(\mathbf{T}^n)$ if

$$\big\|\partial^\alpha f_j-\partial^\alpha f\big\|_{L^\infty(\mathbf{T}^n)}\to 0\quad\text{ as } j\to\infty,\ \forall\,\alpha.$$

Under this notion of convergence, $\mathscr{C}^\infty(\mathbf{T}^n)$ is a topological vector space with topology induced by the family of seminorms $\rho_\alpha(\varphi)=\sup_{x\in\mathbf{T}^n}|(\partial^\alpha f)(x)|$, where α