$$\leq C_p \|b\|_{\mathscr{M}_p} \limsup_{\varepsilon\to 0} \left[\varepsilon^{\frac{n}{2}} \Big\| \sum_{j=1}^{k} \big|P_j L_{\varepsilon/p'}\big| \Big\|_{L^{p'}(\mathbf{R}^n)} \big\|Q L_{\varepsilon/p}\big\|_{L^p(\mathbf{R}^n)} \right]$$

$$= C_p \|b\|_{\mathscr{M}_p} \limsup_{\varepsilon\to 0} \left[\varepsilon^{\frac{n}{2p'}} \Big\| \Big(\sum_{j=1}^{k} \big|P_j\big| \Big) L_{\varepsilon/p'} \Big\|_{L^{p'}(\mathbf{R}^n)} \left(\varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} |Q(x)|^p e^{-\varepsilon\pi|x|^2} dx \right)^{\frac{1}{p}} \right]$$

$$= C_p \|b\|_{\mathscr{M}_p} \Big\| \sum_{j=1}^{k} \big|P_j\big| \Big\|_{L^{p'}(\mathbf{T}^n)} \big\|Q\big\|_{L^p(\mathbf{T}^n)},$$

where we used Hölder's inequality and (4.3.28) in the only inequality above and (4.3.13) in the last equality. Thus we obtain that (4.3.26) implies (4.3.25), and this completes the equivalence of boundedness of $M_b^{\mathscr{F}}$ and $N_b^{\mathscr{F}}$.

We now prove the claimed equivalence for the operators M_b and N_b. We first show that if $M_b^{\mathscr{F}}$ is bounded on $(\mathscr{C}^\infty(\mathbf{T}^n), \|\cdot\|_{L^p})$ with bound independent of the finite set $\mathscr{F}$, then M_b is bounded on $(\mathscr{C}^\infty(\mathbf{T}^n), \|\cdot\|_{L^p})$.

For each $\xi \in \mathbf{R}^n$, let A_ξ be the null subset of $\mathbf{R}^+$ such that $t \mapsto b(\xi/t)$ is continuous on $\mathbf{R}^+ \setminus A_\xi$. We fix a function F in $\mathscr{C}^\infty(\mathbf{T}^n)$, and we note that for each $x \in \mathbf{T}^n$ the function

$$t \mapsto S_{b,t}(F)(x) = \sum_{m\in\mathbf{Z}^n} b(m/t)\widehat{F}(m)e^{2\pi i m\cdot x} \tag{4.3.29}$$

is continuous on the set $\mathbf{R}^+ \setminus \bigcup_{m\in\mathbf{Z}^n} A_m$. We pick a countable dense subset D' of $\mathbf{R}^+ \setminus \bigcup_{m\in\mathbf{Z}^n} A_m$, and we let $D = D' \cup \bigcup_{m\in\mathbf{Z}^n} A_m$. Then D is a countable set and the Lebesgue monotone convergence theorem gives that

$$\Big\| \sup_{t\in D} |S_{b,t}(F)| \Big\|_{L^p(\mathbf{T}^n)} = \lim_{k\to\infty} \big\| M_b^{\mathscr{F}_k}(F) \big\|_{L^p(\mathbf{T}^n)} \leq C_p \|b\|_{\mathscr{M}_p} \big\|F\big\|_{L^p(\mathbf{T}^n)}, \tag{4.3.30}$$

where $\mathscr{F}_k$ is an increasing sequence of finite sets whose union is D. Using that the function in (4.3.29) is continuous on $\mathbf{R}^+ \setminus D$, we conclude that the supremum over $t \in D$ in (4.3.30) can be replaced by the supremum over $t \in \mathbf{Z}^+$ (Exercise 4.3.7).

Assume now that $N_b^{\mathscr{F}}$ is bounded on $(\mathscr{C}_0^\infty(\mathbf{R}^n), \|\cdot\|_{L^p})$ with bound independent of the finite set $\mathscr{F}$. We show that N_b is bounded on $(\mathscr{C}_0^\infty(\mathbf{R}^n), \|\cdot\|_{L^p})$. Let f be in $\mathscr{C}_0^\infty(\mathbf{R}^n)$. We have that the map

$$t \mapsto T_{b,t}(f)(x) = \int_{\mathbf{R}^n} b(\xi/t)\widehat{f}(\xi)e^{2\pi i\xi\cdot x} d\xi = t^n \int_{\mathbf{R}^n} b(\xi)\widehat{f}(t\xi)e^{2\pi i\xi\cdot tx} d\xi \tag{4.3.31}$$

is a continuous function on $\mathbf{R}^+$ since $\widehat{f}$ is continuous. Thus the estimate

$$\Big\| \sup_{t\in D} |T_{b,t}(f)| \Big\|_{L^p(\mathbf{R}^n)} \leq C_p \|b\|_{\mathscr{M}_p} \big\|f\big\|_{L^p(\mathbf{R}^n)} \tag{4.3.32}$$

for a countable dense subset D of $\mathbf{R}^+$ (such as $D = \mathbf{Q}^+$) can be easily extended by replacing the supremum over D by the supremum over $\mathbf{R}^+$. And estimate (4.3.32) for $D = \mathbf{Q}^+$ follows from the corresponding estimate on finite sets via the Lebesgue monotone convergence theorem. $\square$