b is in $\mathscr{M}_p(\mathbf{R}^n)$, since such a conclusion would depend on the values of b on the integer lattice, which is a set of measure zero. However, a converse can be formulated if we assume that for all $R > 0$, the sequences $\{b(m/R)\}_{m\in\mathbf{Z}^n}$ are in $\mathscr{M}_p(\mathbf{Z}^n)$ uniformly in R. Then we obtain that $b(\xi/R)$ is in $\mathscr{M}_p(\mathbf{R}^n)$ uniformly in $R > 0$, which is equivalent to saying that $b \in \mathscr{M}_p(\mathbf{R}^n)$, since dilations of multipliers on $\mathbf{R}^n$ do not affect their norms (see Proposition 2.5.14). These remarks can be precisely expressed in the following theorem.

Theorem 4.3.10. *Suppose that $b(\xi)$ is a bounded function defined on $\mathbf{R}^n$ which is Riemann integrable over any cube. Suppose that the sequences $\{b(\frac{m}{R})\}_{m\in\mathbf{Z}^n}$ are in $\mathscr{M}_p(\mathbf{Z}^n)$ uniformly in $R > 0$ for some $1 < p < \infty$. Then b is in $\mathscr{M}_p(\mathbf{R}^n)$ and we have*

$$\|b\|_{\mathscr{M}_p(\mathbf{R}^n)} \le \sup_{R>0} \big\|\{b(\tfrac{m}{R})\}_{m\in\mathbf{Z}^n}\big\|_{\mathscr{M}_p(\mathbf{Z}^n)}. \tag{4.3.14}$$

Proof. Suppose that f and g are smooth functions with compact support on $\mathbf{R}^n$. Then there is an $R_0 > 0$ such that for $R \ge R_0$, the functions $x \mapsto f(Rx)$ and $x \mapsto g(Rx)$ are supported in $[-1/2, 1/2]^n$. We define periodic functions

$$F_R(x) = \sum_{k\in\mathbf{Z}^n} f(R(x-k)) \quad \text{and} \quad G_R(x) = \sum_{k\in\mathbf{Z}^n} g(R(x-k))$$

on $\mathbf{T}^n$. Observe that the mth Fourier coefficient of F_R is $\widehat{F_R}(m) = R^{-n}\widehat{f}(m/R)$ and that of G_R is $\widehat{G_R}(m) = R^{-n}\widehat{g}(m/R)$.

Now for $R \ge R_0$ we have

$$\begin{aligned}
&\Big|\sum_{m\in\mathbf{Z}^n} b(m/R)\widehat{f}(m/R)\overline{\widehat{g}(m/R)}\ \text{Volume}\left(\tfrac{m}{R}+[0,\tfrac{1}{R}]^n\right)\Big| \qquad (4.3.15)\\
&= \Big|R^n \sum_{m\in\mathbf{Z}^n} b(m/R)\widehat{F_R}(m)\overline{\widehat{G_R}(m)}\Big|\\
&= \Big|R^n \int_{\mathbf{T}^n}\Big(\sum_{m\in\mathbf{Z}^n} b(m/R)\widehat{F_R}(m)e^{2\pi i m\cdot x}\Big)\overline{G_R(x)}\,dx\Big|\\
&\le R^n \big\|\{b(m/R)\}_m\big\|_{\mathscr{M}_p(\mathbf{Z}^n)}\big\|F_R\big\|_{L^p(\mathbf{T}^n)}\big\|G_R\big\|_{L^{p'}(\mathbf{T}^n)}\\
&\le \sup_{R>0}\big\|\{b(m/R)\}_{m\in\mathbf{Z}^n}\big\|_{\mathscr{M}_p(\mathbf{Z}^n)} R^n\big\|F_R\big\|_{L^p(\mathbf{R}^n)}\big\|G_R\big\|_{L^{p'}(\mathbf{R}^n)}\\
&= \sup_{R>0}\big\|\{b(m/R)\}_{m\in\mathbf{Z}^n}\big\|_{\mathscr{M}_p(\mathbf{Z}^n)}\big\|f\big\|_{L^p(\mathbf{R}^n)}\big\|g\big\|_{L^{p'}(\mathbf{R}^n)}. \qquad (4.3.16)
\end{aligned}$$

Since b is bounded and Riemann integrable over any cube in $\mathbf{R}^n$, the function $b(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}$ is Riemann integrable over $\mathbf{R}^n$. The expressions in (4.3.15) are sums associated with the partition $\{\frac{m}{R}+[0,\frac{1}{R}]^n\}_{m\in\mathbf{Z}^n}$ of $\mathbf{R}^n$ which tend to

$$\Big|\int_{\mathbf{R}^n} b(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi\Big|$$