**Lemma 4.3.8.** *Suppose that the function $b$ on $\mathbf{R}^n$ is regulated at the point $x_0$. Let $K_\varepsilon(x) = \varepsilon^{-n}e^{-\pi|x/\varepsilon|^2}$ for $\varepsilon > 0$. Then we have that $(b * K_\varepsilon)(x_0) \to b(x_0)$ as $\varepsilon \to 0$.*

*Proof.* For $r > 0$ define the function

$$F_{x_0}(r) = \frac{1}{r^n}\int_{|t|\le r}\big(b(x_0-t)-b(x_0)\big)\,dt = \frac{1}{r^n}\int_0^r\int_{\mathbf{S}^{n-1}}\big(b(x_0-s\theta)-b(x_0)\big)\,d\theta s^{n-1}ds.$$

Let $\eta > 0$. Since $b$ is regulated at $x_0$ there is a $\delta > 0$ such that for $r \le \delta$ we have $|F_{x_0}(r)| \le \eta$. Fix such a $\delta$ and write

$$(b * K_\varepsilon)(x_0) - b(x_0) = \int_{y\in\mathbf{R}^n}\big(b(x_0-y)-b(x_0)\big)K_\varepsilon(y)\,dy = A_1^\varepsilon + A_2^\varepsilon,$$

where

$$A_1^\varepsilon = \int_{|y|\ge\delta}\big(b(x_0-y)-b(x_0)\big)K_\varepsilon(y)\,dy$$

and

$$\begin{aligned}
A_2^\varepsilon &= \int_{|y|<\delta}\big(b(x_0-y)-b(x_0)\big)K_\varepsilon(y)\,dy \\
&= \int_0^\delta \frac{1}{\varepsilon^n}e^{-\pi(r/\varepsilon)^2}\int_{\mathbf{S}^{n-1}}\big(b(x_0-r\theta)-b(x_0)\big)\,d\theta r^{n-1}dr \\
&= \int_0^\delta \frac{1}{\varepsilon^n}e^{-\pi(r/\varepsilon)^2}\frac{d}{dr}\big(r^nF_{x_0}(r)\big)\,dr\,.
\end{aligned}$$

For our given $\eta > 0$ there is an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ we have

$$|A_1^\varepsilon| \le 2\|b\|_{L^\infty}\int_{|y|\ge\frac{\delta}{\varepsilon}} e^{-\pi|y|^2}dy < \eta\,.$$

Via an integration by parts $A_2^\varepsilon$ can be written as

$$\begin{aligned}
|A_2^\varepsilon| &= \left|\delta^nF_{x_0}(\delta)\frac{1}{\varepsilon^n}e^{-\pi(\delta/\varepsilon)^2} - 0 + 2\pi\int_0^\delta \frac{r}{\varepsilon^{n+2}}e^{-\pi(r/\varepsilon)^2}r^nF_{x_0}(r)\,dr\right| \\
&= \left|F_{x_0}(\delta)\frac{\delta^n}{\varepsilon^n}e^{-\pi(\delta/\varepsilon)^2} + 2\pi\int_0^{\delta/\varepsilon} r^{n+1}F_{x_0}(\varepsilon r)e^{-\pi r^2}dr\right| \\
&\le \big|F_{x_0}(\delta)\big|\frac{\delta^n}{\varepsilon^n}e^{-\pi(\delta/\varepsilon)^2} + \sup_{0<r\le\frac{\delta}{\varepsilon}}|F_{x_0}(\varepsilon r)|2\pi\int_0^{\delta/\varepsilon} r^{n+1}e^{-\pi r^2}dr \\
&\le \big|F_{x_0}(\delta)\big|C_n + \sup_{0<r\le\delta}|F_{x_0}(r)|C_n' \\
&\le (C_n + C_n')\eta\,,
\end{aligned}$$

where we set $C_n = \sup_{t>0}t^ne^{-\pi t^2}$ and $C_n' = 2\pi\int_0^\infty r^{n+1}e^{-\pi r^2}dr$. Then for $\varepsilon < \varepsilon_0$ we have $|(b*K_\varepsilon)(x_0 - b(x_0)| < (C_n + C_n' + 1)\eta$, thus $(b*K_\varepsilon)(x_0) \to b(x_0)$ as $\varepsilon \to 0$. $\square$